FINITE ELEMENT METHOD II
lecture notes

I. Error estimation and adaptive remeshing.
II. Steady state heat flow and thermal stresses.
III. Introduction to structural dynamics, free vibrations.
IV. Nonlinear problems in mechanics of structures - basic numerical techniques.
V. Parametric modeling and design optimization

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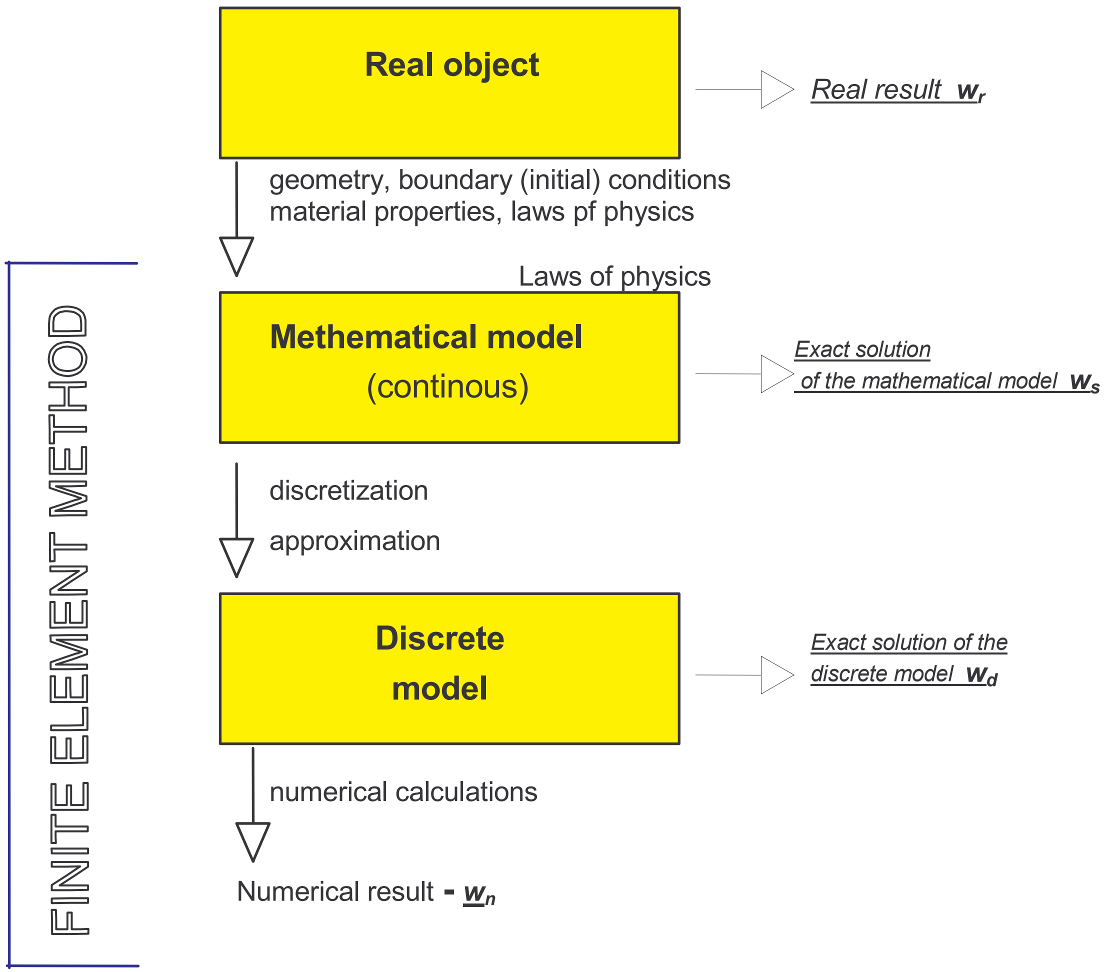
1. ACCURACY OF FE ANALYSIS. ERROR ESTIMATION



$\varepsilon_s = w_s - w_r$ **– modeling error,**

$\varepsilon_d = w_d - w_s$ **– discretization error**

$\varepsilon_n = w_n - w_d$ **– numerical error**

Total error:

$\varepsilon_c = \varepsilon_s + \varepsilon_d + \varepsilon_n = w_n - w_r$

The 'best' FEM analysis: $\varepsilon_s \sim \varepsilon_d \sim \varepsilon_n$

Modeling error ε_s

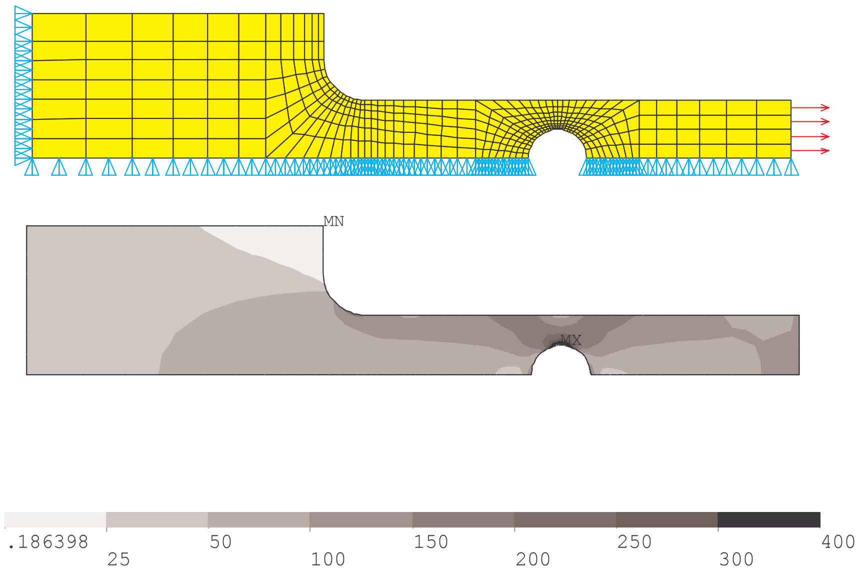
depends on the accuracy of the available information about the problem and knowledge of the analyst (1D – 2D – 3D models, linear, nonlinear , assumed simplifications …..)

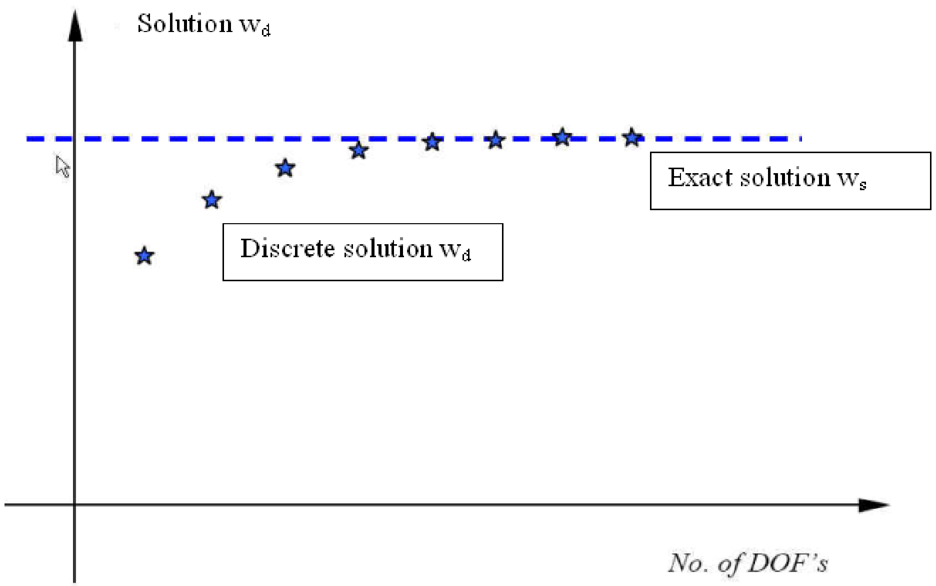
Approximate methods – flow chart

(Boundary Element Method - BEM, Finite Differences Method - FDM, Finite Element Method – FEM and others)

Discretization error ε_d

Depends on the mesh density, types of the elements – shape functions, the shapes of the elements (regular polygons, regular solids),





FE mesh and von Mises stress distribution

Discrete solution versus exact solution

Numerical error ε_n

$$[K]\{q\}=\{F\},$$

$$[K+\delta K]\{q+\delta q\}=\{F+\delta F\}. \qquad \longrightarrow \qquad \frac{\|\delta q\|}{\|q\|}\le \|[K]\|\|[K]^{-1}\|\left(\frac{\|\{\delta F\}\|}{\|\{F\}\|}+\frac{\|[\delta K]\|}{\|[K]\|}\right),$$

condition number of the matrix K

$$cond([K])=\|[K]\|\,\|[K]^{-1}\|.$$

Vector or matrix norm – a measure of magnitude

L_2- Euclidean norms

Vector norm $$\|\{q\}\|=\left(\sum_i (q_i)^2\right)^{\frac{1}{2}},$$

Matrix norm $$\|[K]\|=\left(\sum_j\sum_i (k_{ij})^2\right)^{\frac{1}{2}}$$

Max norms ($L_\sim$ norms)

$$\|\{q\}\|=\max_i |q_i|, \qquad \|[K]\|=\max_i\left(\sum_j |k_{ij}|\right).$$

$\| K \| \ge 1$,

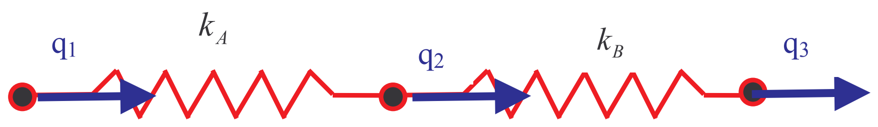
$\| K \| \approx 1$ - problem well-conditioned

$\| K \| \gg 1$ - problem ill-conditioned

Reasons of ill-conditioning of the problems in FE analysis – great differences in stiffness of FE elements, unstable boundary conditions (great sensitivity)

Example

The example (ill conditioned problem):



$$[k]_e = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}_{2\times 2} \qquad \begin{bmatrix} k_A & -k_A & \\ -k_A & k_A + k_B & -k_B \\ & -k_B & k_B \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$q_1 = 0 \qquad \begin{bmatrix} k_A + k_B & -k_B \\ -k_B & k_B \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

Let's assume: $k_A = 1 \qquad k_B = 1000$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

The solution:

$$\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{k_A} & \frac{1}{k_A} \\ \frac{1}{k_A} & \frac{1}{k_A} + \frac{1}{k_B} \end{bmatrix} \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

The result:

$$\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.001 \end{Bmatrix}$$

First equation

For disturbed force vector : $\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0.999 \\ -1.0 \end{Bmatrix}$

$$\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} -0.001 \\ -0.002 \end{Bmatrix}$$

$$\frac{\|\delta F\|}{\|F\|} \approx 1 \cdot 10^{-3} \quad \Rightarrow \quad \frac{\|\delta q\|}{\|q\|} \approx 1$$

$$q_3 = \frac{k_A + k_B}{k_B} \cdot q_2 - \frac{F_2}{k_B}$$

Second equation

$$q_3 = q_2 + \frac{F_3}{k_B}$$

System ill-conditioned if

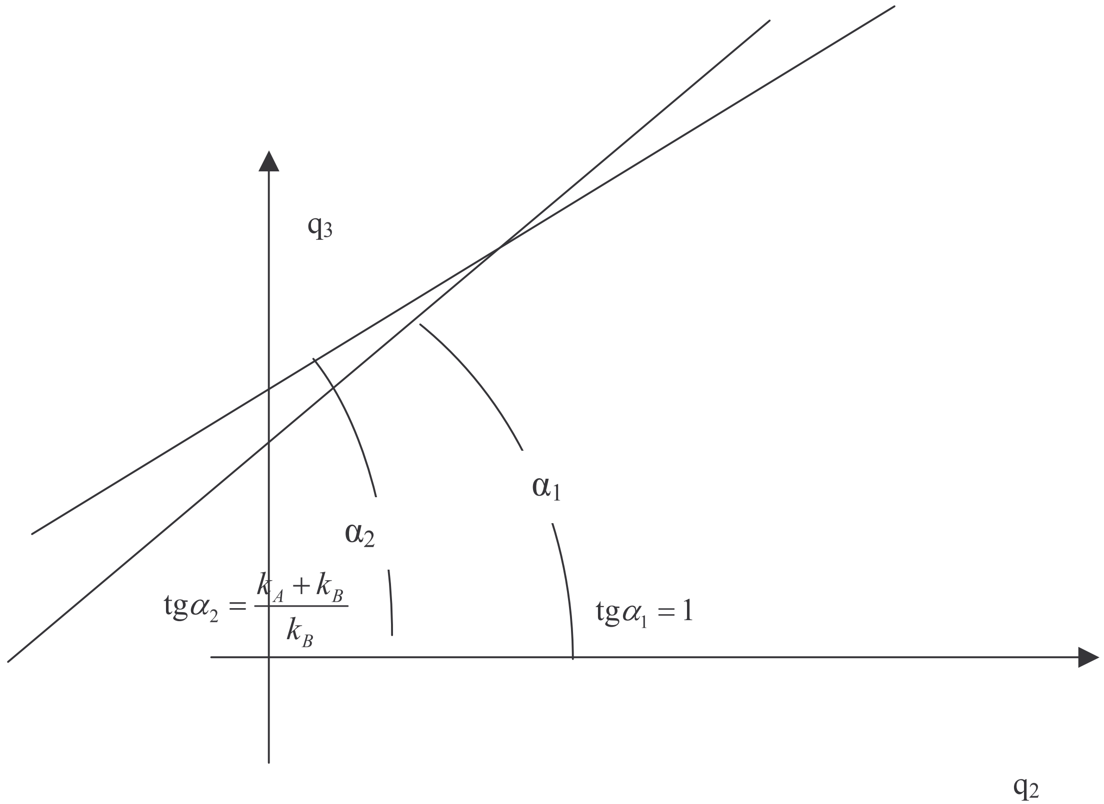
$$\frac{k_A + k_B}{k_B} \to 1$$

$$\frac{k_A}{k_B} \to 0$$

Sensitivity to the slop change

$K_A << K_B$ **system ill conditioned**

$K_A > K_B$ **system well conditioned**



Round – off terror

$$r \geq p - \log_{10}\left(cond\left(\left[K\right]\right)\right)$$

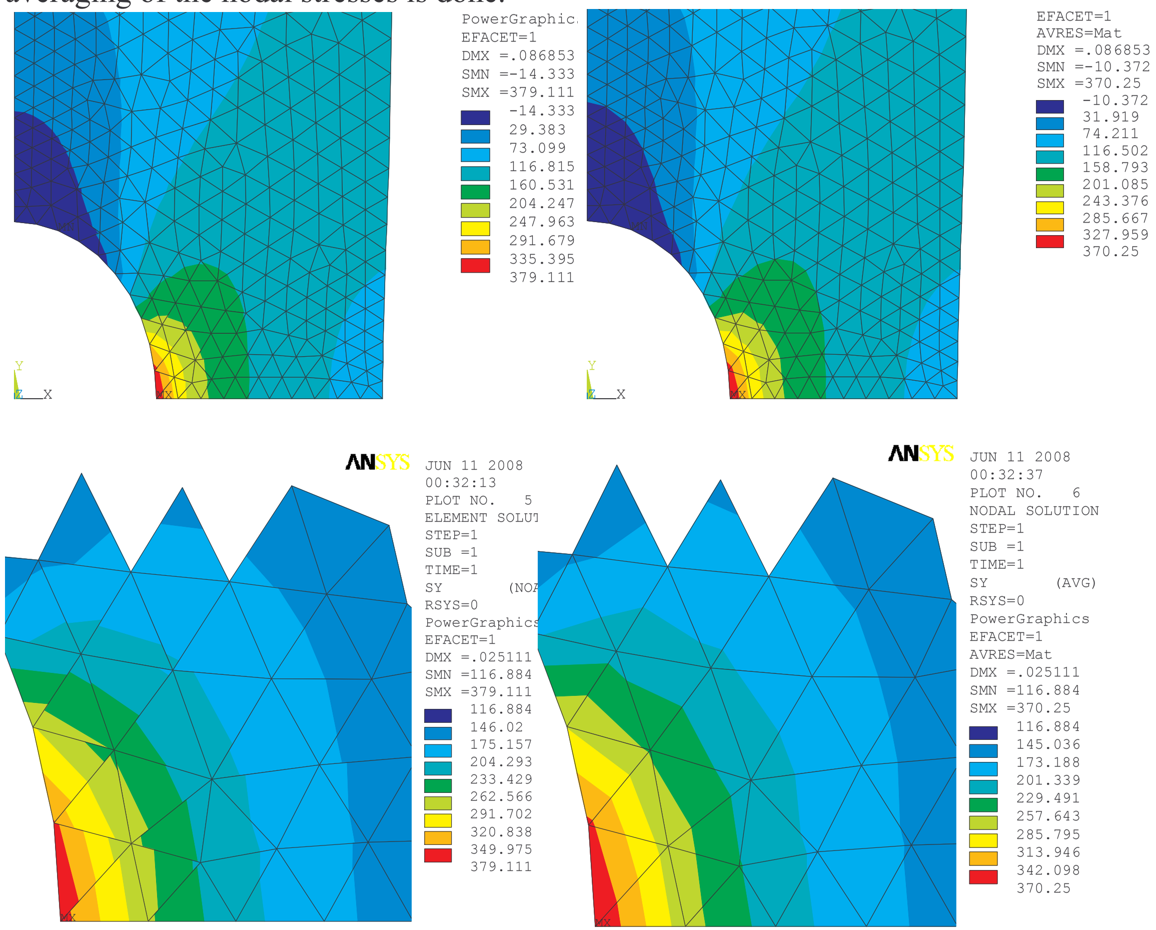
p – number of significant digits in the computer representation of numbers

r – number of significant digits of the result

A posteriori error approximation techniques

Element and nodal solution in FE program postprocesor (PLESOL, PLNSOL in ANSYS)

FE solution results in a continuous displacement field (from element to element), and a discontinuous stress field. To obtain more acceptable stresses, averaging of the nodal stresses is done.



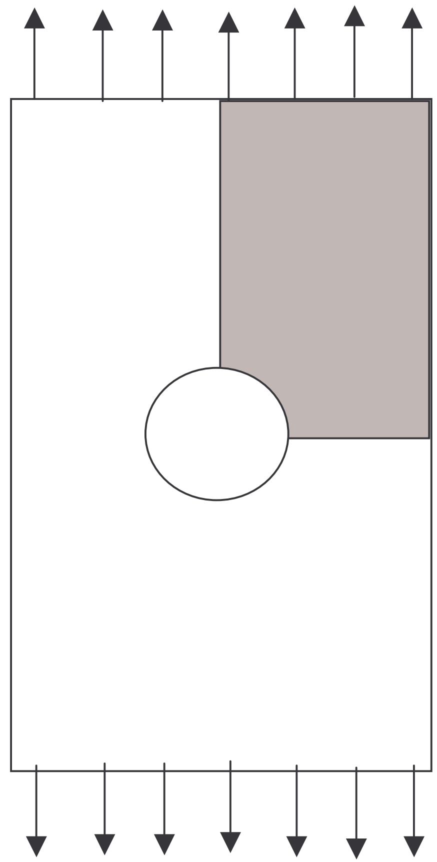
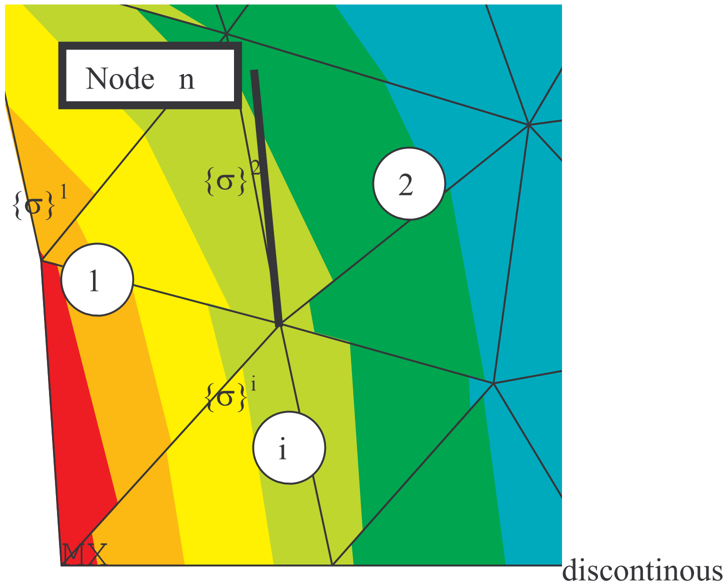
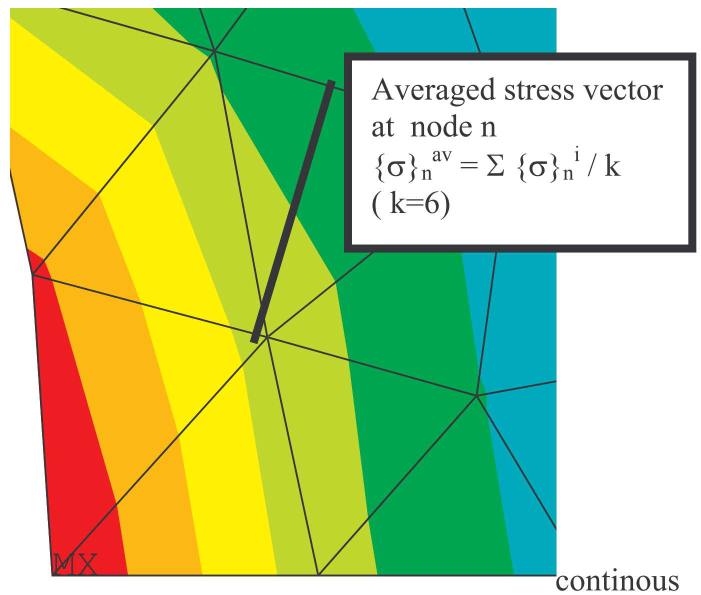


Plate with a hole under tension. Quarter model. „Element solution" (left) and " nodal solution" (right). Six-node triangular plane elements





stress vector at node n of element i

$$\{\sigma\}_n^i = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}_n^i \quad \{\sigma\}_n^1 \neq \{\sigma\}_n^2 \neq \{\sigma\}_n^3 \neq \ldots$$

averaged stress vector at node n: $$\{\sigma\}_n^{av} = \frac{\sum_{i=1}^{k} \{\sigma\}_n^i}{k}$$

stress error vector at node n of element *i*

$$\{\Delta\sigma\}_n^i = \{\sigma\}_n^i - \{\sigma\}_n^{av}$$

The stress error vector $\{\Delta\sigma\}^i$ within the element ' i' may be determined by standard approximation using the stress error vectors at nodes of element i $\{\Delta\sigma\}_n^i$

strain energy of the element i

$$U_i = \frac{1}{2}\int_{\Omega_i} \lfloor\sigma\rfloor^i \{\varepsilon\}^i \, d\Omega_i$$

$\{\varepsilon\} = [D]^{-1}\{\sigma\}$, D] – stress-strain matrix

$$U_i = \frac{1}{2}\int_{\Omega_i} \lfloor\sigma\rfloor^i [D]^{-1} \{\sigma\}^i \, d\Omega_i$$

Then for each element so called energy error can be estimated

$$e_i = \frac{1}{2}\int_{\Omega_i} \lfloor\Delta\sigma\rfloor^i [D]^{-1} \{\Delta\sigma\}^i \, d\Omega_i \qquad \text{(ETABLE-SERR)}$$

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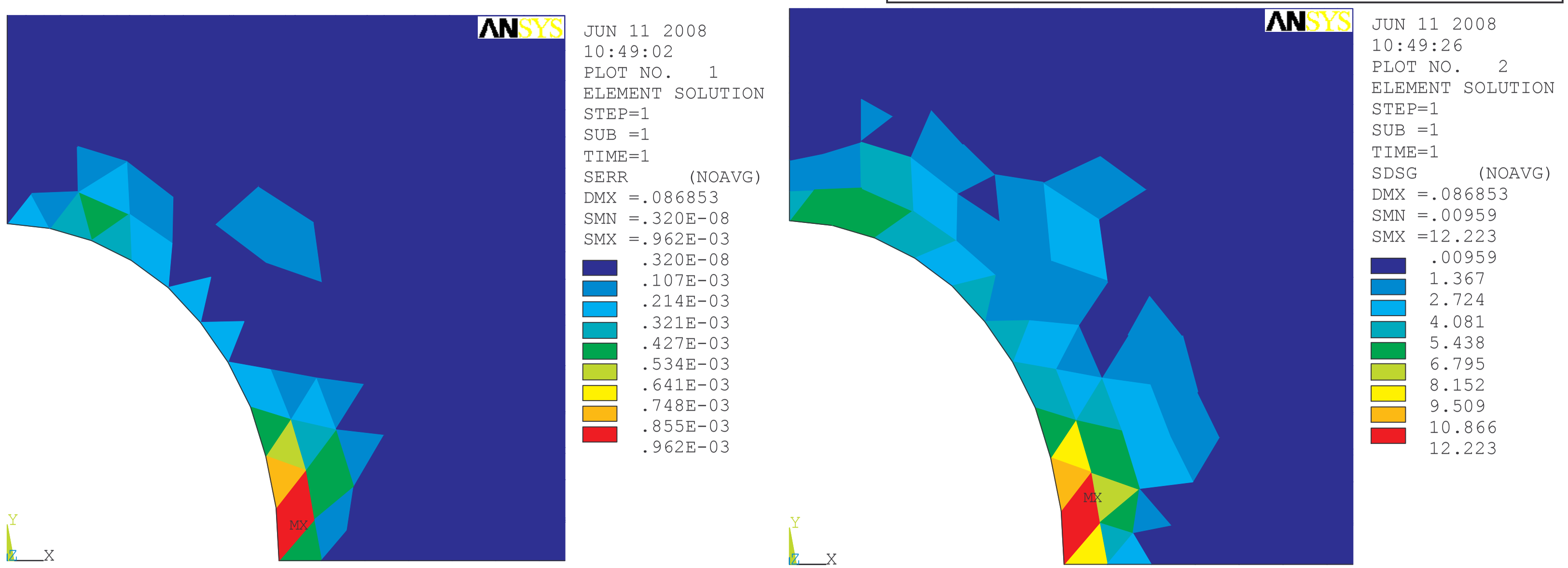


Initial mesh. SERR and SDSG error distribution

SDSG - $\Delta\sigma_i$ = maximum absolute value of any component of $\{\Delta\sigma\}_n^i = \{\sigma\}_n^i - \{\sigma\}_n^{av}$ *for all nodes connected to element*

The energy error over the model

$$e = \sum_{i=1}^{l.el.} e_i$$

The energy error can be normalized against the strain energy

$$\mathrm{SEPC} = 100\left(\frac{e}{U+e}\right)^{\frac{1}{2}}$$

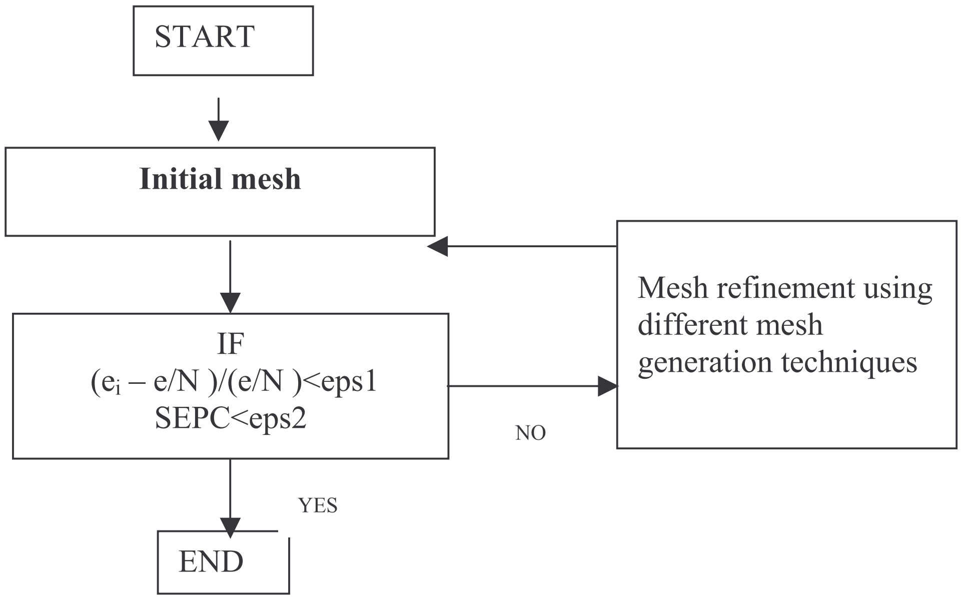
U – total strain energy over the entire model

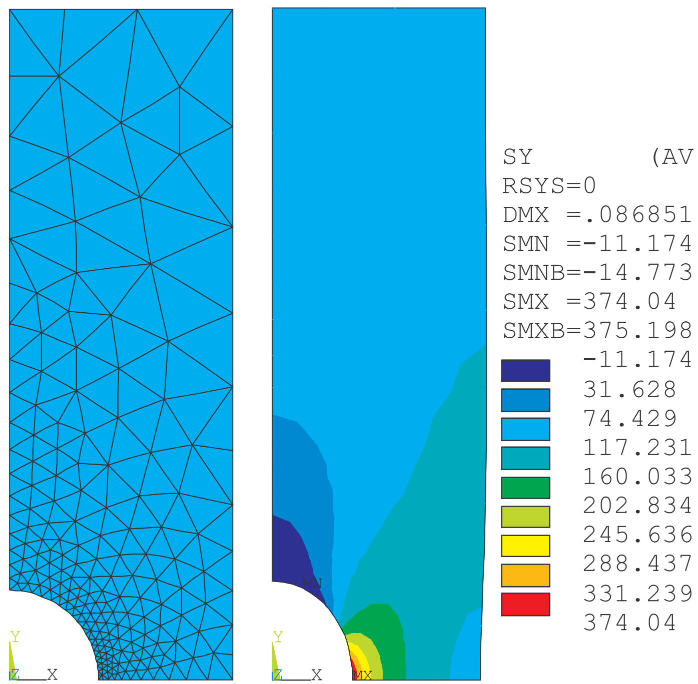
SEPC – percentage error in energy norm

The e_i values can be used for adaptive mesh refinement. It has been shown that if e_i is equal for all elements, then the model using the given number of elements is the most efficient one. This concept is also referred to as "error equilibration" ($e_i = \mathrm{const}$,SEPC< S_0).

Adaptive Meshing Techniques

- Automatic refinement of FE meshes until converged results are obtained
- User's responsibility reduced: only need to generate a good initial mesh





Final FE mesh and the results - Sy distribution (percentage error in energy norm *SEPC=0.811%, uniform error distribution* e_i*= const)*

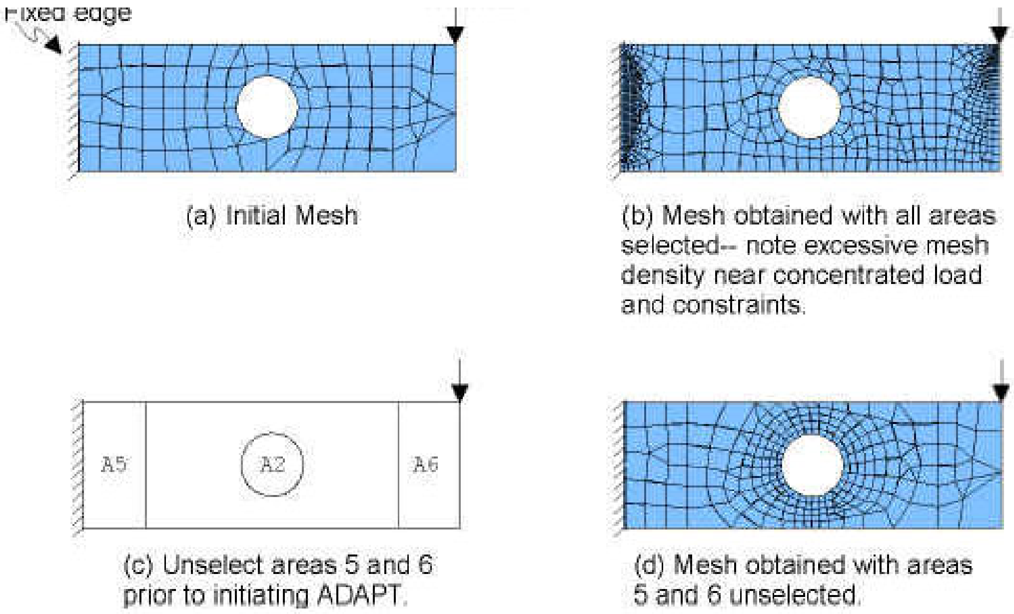
Application – e.g. in metal forming and high-speed impact analyses when a body may experience very large amounts of plastic deformation. The solution may give inaccurate results in these situations due to inadequate element aspect ratios.

Requirements for Adaptive Meshing in ANSYS (prewritten macro ADAPT.MAC to perform adaptive meshing)

- The procedure is valid only for linear static structural and linear steady-state thermal analyses.
- Use only *one* material type, as the error calculations are based in part on *average* nodal stresses
- Use *meshable* solid model entities. (Avoid characteristics that will cause meshing failure)

Selective adaptive meshing

If mesh discretization error (measured as a percentage) is relatively unimportant in some regions of the model (for instance, in a region of low, slowly-changing stress), the procedure may be speed up by excluding such regions from the adaptive meshing operations. Also - near singularities caused by concentrated loads and at boundaries between different materials.



(a) Initial Mesh

(b) Mesh obtained with all areas selected-- note excessive mesh density near concentrated load and constraints.

(c) Unselect areas 5 and 6 prior to initiating ADAPT.

(d) Mesh obtained with areas 5 and 6 unselected.

Selective adaptive meshing in ANSYS

Types of refinement in adaptive meshing :

h-refinement: reduction the size of the element (“*h*” refers to the typical size of the elements)

p-refinement: increase the order of the polynomials on an element (shape functions from linear to quadratic, etc.)

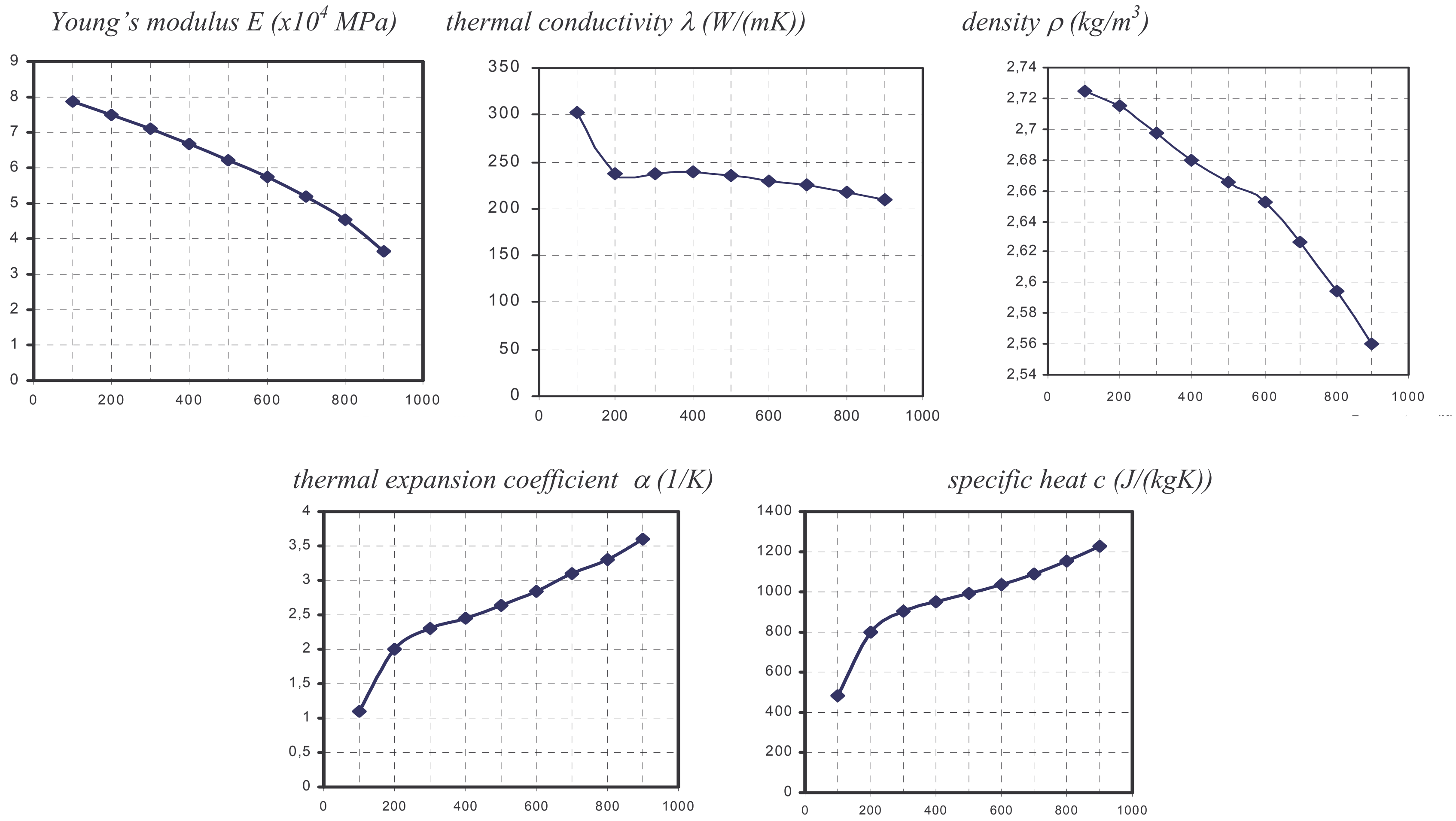
r-refinement: re-arrangement the nodes in the mesh

hp-refinement: combination of the h- and p-refinements.

2. HEAT TRANSFER AND THERMAL STRESSES

Temperature may influence the strength of a structure by:

- thermal expansion effect (thermal stresses)
- impact of the temperature on the mechanical properties of materials



Thermo-mechanical properties of aluminium (properties vs temperature K)

Thermal expansion coefficient

$$\alpha(T)=\frac{l(T)-l(T_{ref})}{(T-T_{ref})\,l(T_{ref})}=\frac{\Delta l}{l\,\Delta T}=\frac{\varepsilon(T)}{\Delta T}$$

$$\{\varepsilon\}=\{\varepsilon\}_T+\{\varepsilon\}_s \qquad \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=\begin{Bmatrix}\alpha_x\Delta T\\ \alpha_y\Delta T\\ \alpha_z\Delta T\\ 0\\ 0\\ 0\end{Bmatrix}_T+\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}_s$$

T_0 – reference temperature for an unstrained state, in isotropic case $\alpha_x=\alpha_y=\alpha_z=\alpha$

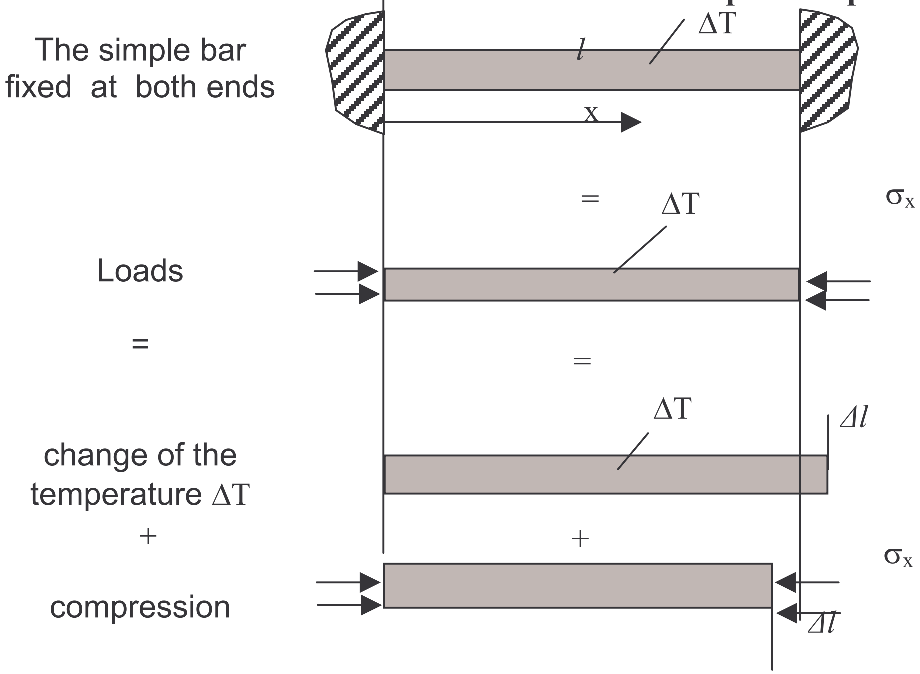
Thermal stresses may be caused by

- non uniform temperature field
- nonhomogeneous materials
- statically indeterminant constraints

Total strain vector is the sum of thermal strain and elestic strain vectors.

(Hooke's law $\{\sigma\}=[D]\{\varepsilon\}_s$!)

Example – simple relation temperature-stress



The elongation equals 0 → relative elongation and longitudinal strain = 0

$$\varepsilon_x=\varepsilon_{xT}+\varepsilon_{xs}=0\,,$$

$$\varepsilon_{xT}=\alpha\Delta T\,,\ \varepsilon_{xs}=\sigma_x/E\,.$$

$$\varepsilon_{xs}=-\varepsilon_{xT}=-\alpha\Delta T$$

$$\sigma_x=E\varepsilon_{xs}=-E\alpha\Delta T$$

For typical steel ($E=2\times10^5$ MPa, $\nu=0,3$, $\alpha=1,2\times10^{-5}\ 1/°C$) and $\Delta T=100°C$ $\sigma_x=240$ MPa

THERMAL ANALYSIS

Partial differential equation describing transient heat flow through a solid (law of conservation of energy):

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) + q_v(x,y,z,t),$$

$T(x,y,z,t)$ – temperature,

q_v – internal heat generation rate per unit volume(W/m^3),

$\lambda_x, \lambda_y, \lambda_z$ – heat conductivity coefficients(W/mK),

ρ – density (kg/m^3),

c – specific heat (J/kg).

$$\frac{\partial T}{\partial t} = a_d \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = a_d \nabla^2 T,$$

where $\alpha_d = \lambda / c\rho$ is the thermal diffusivity

Thermal properties of selected materials at 20°C (RT)

Material	Thermal expansion coefficient α (1/^{0}C)	Thermal conductivity λ (W/mK)	Specific heat c (J/kgK)	Density ρ (kg/m^3)
Copper	$1{,}7 \cdot 10^{-5}$	390	400	9000
Aluminium	$2{,}4 \cdot 10^{-5}$	210	900	2700
Pine wood	$0{,}4–0{,}6 \cdot 10^{-5}$	0,1–0,5	1300–2700	500–700
Steel 1H13	$1{,}1 \cdot 10^{-5}$	29	440	7700
Glass	$0{,}05–0{,}09 \cdot 10^{-5}$	0,7–1,3	600–800	2500
Rubber	$7{,}7 \cdot 10^{-5}$	0,16	1400	1200

Convective Heat Flow

The rate of heat flow across a boundary is proportional to the difference between the surface temperature and the temperature of adjacent fluid

$$q = \alpha_k (T_0 - T_c)$$

where α_k is the convection coefficient (film coefficient)

Typical magnitudes of convection coeeficient (W/(m^{2}K))

Medium (fluid)	Free convection	Forced convection
gas (air)	5–30	30–500
water	30–300	300–20000
oil	5–100	30–3000
liquid metals	50–500	500–20000

Conduction

Is the transfer of thermal energy through the solid or fluid due to temperature gradient.

The equation describing this heat transfer is Fourier's law.

For an isotropic medium: $\overline{q} = -\lambda \operatorname{grad}(T)$

where $\overline{q}$ is the rate of heat flow per unit area and λ is the thermal conductivity.

Radiation Heat Exchange

The rate of heat flow cross a boundary in the form of thermal radiation .

Stefan – Boltzmann law:

$$e = \varepsilon\sigma_0 T^4 = CT^4,$$

$$\sigma_0 = 5{,}67 \cdot 10^{-8}\ \mathrm{W/m^2K^4}$$

ε emissivity of the surface ($0 < \varepsilon < 1$)

The heat exchange between two parallel surfaces

$$q_{AB} = \varepsilon_{AB} C_o \left[(T_A/100)^4 - (T_B/100)^4 \right]$$

$$C_o = 10^8\,\sigma_0,$$

$$\varepsilon_{AB} = \frac{1}{1/\varepsilon_A + 1/\varepsilon_B - 1}$$

In computetional practice heat exchange across boundary (by radiation and convection) is described by convection model $q = \alpha_k (T_0 - T_c)$ where α_k is adequate function of temperature.

The simplest case of heat flow – steady state heat transfer, constant isotropic material properties. In that case the heat flow equation reduces to Poisson's equation:

$$\nabla^2 T + \mathrm{f} = 0$$

FE method for Poisson's equation in 2D space

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+f(x_1,x_2)=0,$$

where $\overline{x}=(x_1,x_2)\in\Omega$, u

Boundary conditions

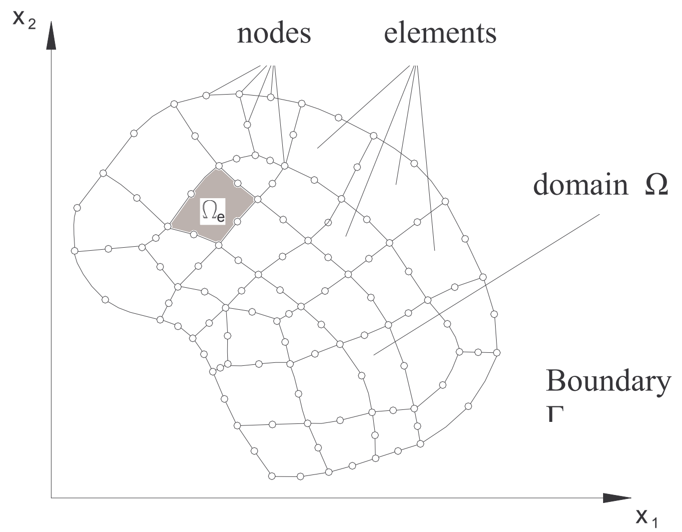
$$u(\overline{x})=u_0 \quad , \quad \overline{x}\in\Gamma_u$$

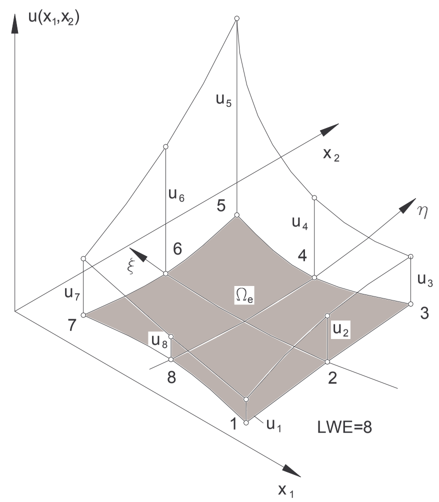
$$q(x)=\frac{\partial u(\overline{x})}{\partial n}=q_0 \quad , \quad \overline{x}\in\Gamma_q$$

(prescribed temperature or prescribed thermal flux)

Minimized functional in FEM formulation

$$I(u)=\frac{1}{2}\int_{\Omega}\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2-2f(x_1,x_2)u\right]d\Omega-\int_{\Gamma_q}q_0ud\Gamma,$$





$$\Omega=\bigcup_{i=1}^{LE}\Omega_e \quad i$$

LE – number of the elements in the domain Ω

Approximation of the unknown function (temperature) within an element

u_i , i = 1,...,LWE nodal temperatures,

$$u(x_1,x_2)=\sum_{i=1}^{LWE}N_i(x_1,x_2)u_i$$

$N_i(x_1,x_2)$ – shape functions.

LWE – number of nodes of the element,

$$I(u) \cong \sum_{i=1}^{LE} \frac{1}{2} \int_{\Omega_i} \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 - 2 f(x_1, x_2) u \right] d\Omega_i - \sum_{j=1}^{LK} \int_{\Gamma_j} q_0 u \, d\Gamma_j$$

LK number of element sides on Γ_q .

Within elements:

$$\frac{\partial u}{\partial x_1} = \sum_{i=1}^{LWE} \frac{\partial N_i}{\partial x_1} u_i,$$

$$\frac{\partial u}{\partial x_2} = \sum_{i=1}^{LWE} \frac{\partial N_i}{\partial x_2} u_i.$$

Finally the mimimized functional is replaced by the function of several variables u_i

$$I(u) \approx \frac{1}{2} \lfloor u_1, u_2, u_3, \ldots, u_{LW} \rfloor \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1LW} \\ k_{21} & k_{22} & k_{23} & & \\ k_{31} & k_{32} & & & \\ \cdots & & & & \\ k_{LW1} & & & & k_{LW\,LW} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdots \\ u_{LW} \end{Bmatrix} - \lfloor u_1, u_2, u_{3,} \ldots, u_{LW} \rfloor \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \\ b_{LW} \end{Bmatrix}$$

$$I \approx \frac{1}{2} \underset{1\times LW}{\lfloor u \rfloor} \underset{LW\times LW}{[K]} \underset{LW\times 1}{\{u\}} - \underset{1\times LW}{\lfloor u \rfloor} \underset{LW\times 1}{\{b\}}.$$

Necessary (sufficient) condition:

$$\frac{\partial I}{\partial u_i} = 0, \qquad i = 1, \ldots, LW. \qquad \Rightarrow \qquad [K]\{u\} = \{b\}$$ **+ Dirichlet b.c.**

Thermal stresses – FE equation

$$\{\varepsilon\} = \{\varepsilon\}_T + \{\varepsilon\}_s ,$$

$$\{\varepsilon\}_s = [D]^{-1}\{\sigma\} .$$

[D] – material stiffness matrix $[D]^{-1}$ material flexibility matrix

$$\{\varepsilon\} = [B]\{q\}_e .$$

[B] – element strain matrix

: $$\{\sigma\} = [D]\{\varepsilon\}_s = [D]\left(\{\varepsilon\} - \{\varepsilon\}_T\right) = [D]\left([B]\{q\}_e - \{\varepsilon\}_T\right). \quad \textbf{(*)}$$

FEM equation derived from the principle of virtual work

$\{\delta q\}_e$ - vector of nodal virtual displacements of an element

$\{\delta\varepsilon\} = [B]\{\delta q\}_e$ - vector of virtual strains within the element

Principle of virtual work for an element

$$\lfloor \delta q \rfloor_e \{F\}_e = \int_{\Omega_e} \lfloor \delta\varepsilon \rfloor \{\sigma\}\, d\Omega_e .$$

$$\lfloor \delta q \rfloor_e \{F\}_e - \lfloor \delta q \rfloor_e \int_{\Omega_e} [B]^T \{\sigma\}\, d\Omega_e = 0,$$

$$\lfloor \delta q \rfloor_e \left(\{F\}_e - \int_{\Omega_e} [B]^T \{\sigma\}\, d\Omega_e \right) = 0.$$

$$\{F\}_e - \int_{\Omega_e} [B]^T \{\sigma\}\, d\Omega_e = 0$$

Using Hooke's law (*) we have:

$$[k]_e \{q\}_e = \{F\}_e + \{F_T\}_e ,$$

$[k]_e = \int_{\Omega_e} [B]^T [D][B]\, d\Omega_e$ - element stiffness matrix

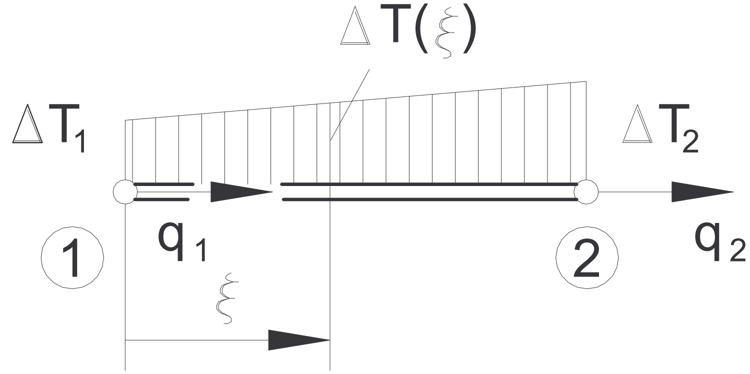
$\{F_T\}_e = \int_{\Omega_e} [B]^T [D]\{\varepsilon\}_T\, d\Omega_e$ - additional vector of nodal forces

(nodal forces caused by temperatures)

Thermal stresses in rod elements

Basic relations for 2 node rod element

$$u(\xi) = \lfloor N_1, N_2 \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e , \qquad N_1(\xi) = 1 - \frac{\xi}{l_e}, \qquad N_2(\xi) = \frac{\xi}{l_e}.$$



$$\varepsilon(\xi) = \frac{du}{d\xi} = \lfloor N_1', N_2' \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e , \qquad \sigma(\xi) = E\left(\varepsilon(\xi) - \varepsilon_T\right).$$

$$\Delta T(\xi) = \lfloor N_1(\xi), N_2(\xi) \rfloor \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix}, \quad \{F_T\}_e = \int_{\Omega_e} [B]^T [D] \{\varepsilon\}_T \, d\Omega_e$$

Vector of thermal nodal forces

$$\{F_T\}_e = \int_{\Omega_e} [B]^T [D] \{\varepsilon\}_T \, d\Omega_e = \int_{\Omega_e} \begin{Bmatrix} N_1' \\ N_2' \end{Bmatrix} \cdot \alpha \cdot E \lfloor N_1, N_2 \rfloor d\Omega_e \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix} =$$

$$= \alpha EA \int_0^{l_e} \begin{bmatrix} N_1'N_1 & N_1'N_2 \\ N_2'N_1 & N_2'N_2 \end{bmatrix} d\xi \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix} = \alpha EA \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix},$$

$$\{F_T\}_e = \frac{\Delta T_1 + \Delta T_2}{2} \alpha EA \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

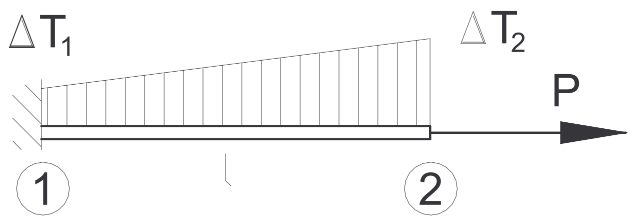
$$[B]^T = \lfloor N_1', N_2' \rfloor^T = \begin{Bmatrix} N' \\ N_2' \end{Bmatrix},$$

$$[D] = E,$$

$$\{\varepsilon\}_T = \varepsilon_T = \alpha \Delta T(\xi) = \alpha \lfloor N_1, N_2 \rfloor \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix}$$

EXAMPLE

Find the elongation of the rod loaded by the force P and the temperature distribution ΔT



$$[k]_e \{q\}_e = \{F\}_e + \{F_T\}_e ,$$

$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \end{Bmatrix} + \frac{\Delta T_1 + \Delta T_2}{2}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\cdot \alpha EA .$$

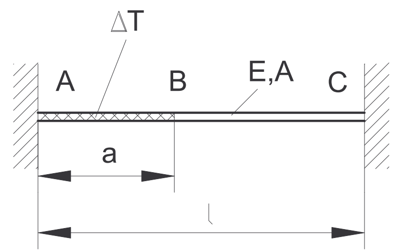
$$q_1 = 0$$

$$\frac{EA}{l}\cdot q_2 = P + \frac{\Delta T_1 + \Delta T_2}{2}\alpha EA ,$$

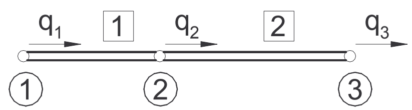
$$q_2 = \alpha\Delta T\frac{l_1 l_2}{l_1 + l_2} = \alpha\Delta T\frac{a(l-a)}{l} ,$$

EXAMPLE

Find the stresses in both part of the constrained with heated part AB



$$EA\begin{bmatrix} \frac{1}{l_1} & -\frac{1}{l_1} & 0 \\ -\frac{1}{l_1} & \frac{1}{l_1}+\frac{1}{l_2} & -\frac{1}{l_2} \\ 0 & -\frac{1}{l_2} & \frac{1}{l_2} \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ \alpha EA\Delta T \\ F_3 \end{Bmatrix} ,$$



$l_1 = a$, $l_2 = l - a$

$q_1 = 0, \; q_3 = 0$ =>

$$EA\left(\frac{l_1 + l_2}{l_1 l_2}\right) q_2 = \alpha EA\Delta T .$$

$$q_2 = \alpha\Delta T\frac{l_1 l_2}{l_1 + l_2} = \alpha\Delta T\frac{a(l-a)}{l} ,$$

Element 1 :

$$\varepsilon_1 = \lfloor N_1', N_2' \rfloor\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \left\lfloor -\frac{1}{a}, \frac{1}{a} \right\rfloor\begin{Bmatrix} 0 \\ q_2 \end{Bmatrix} = \frac{q_2}{a} = \alpha\Delta T\frac{l-a}{l} ,$$

$$\sigma_1 = E(\varepsilon_1 - \alpha\Delta T) = -\alpha\Delta TE\frac{a}{l} .$$

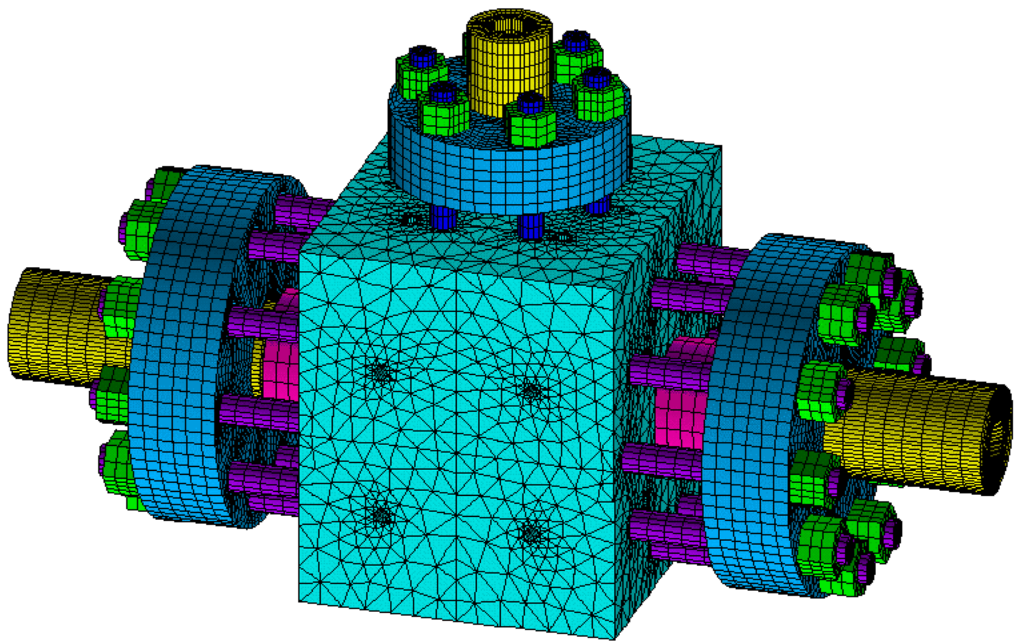
Element 2:

$$\varepsilon_2 = \lfloor N_1', N_2' \rfloor\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \left\lfloor \frac{-1}{l-a}, \frac{1}{l-a} \right\rfloor\begin{Bmatrix} q_2 \\ 0 \end{Bmatrix} = \frac{-q_2}{l-a} = -\alpha\Delta T\frac{a}{l} ,$$

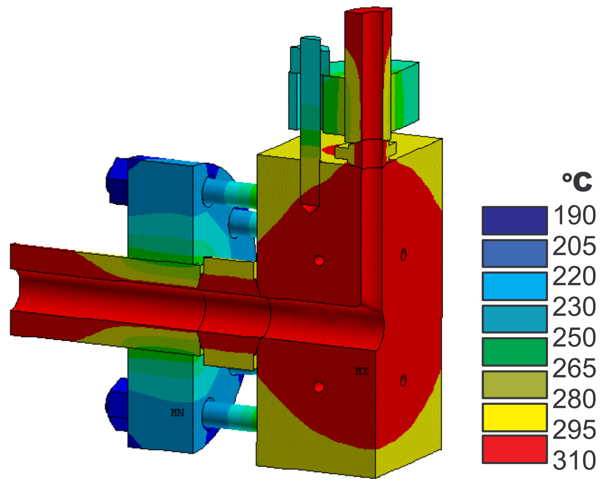
$$\sigma_2 = E(\varepsilon_2 - 0) = -\alpha\Delta TE\frac{a}{l} .$$

FE analysis of a high pressure T-connection

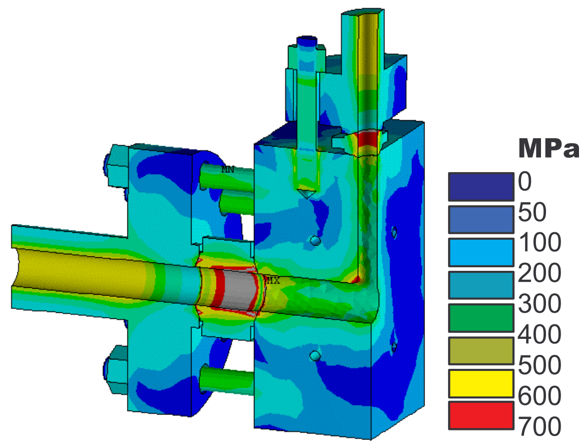
The aim of the analysis was to find stress and strain distribution in a T-connection caused by high internal pressure (2600 at) and temperature gradients. External cooling, assembly procedure (screw pretension), contact and plasticity effects have been included. The project done for ORLEN petrochemical company



FE model



Temperature distribution



Von Mises stress

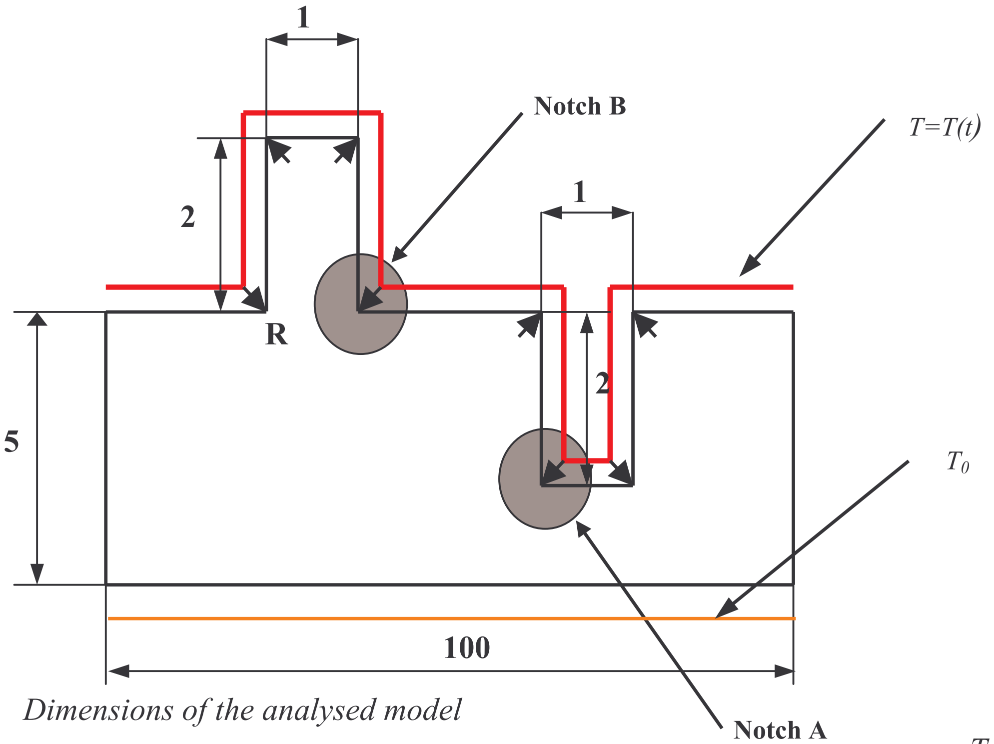
FEM ANALYSIS OF LOCAL STRESS CONCENTRATIONS CAUSED BY IMPACT THERMAL LOADS

Analysis description:

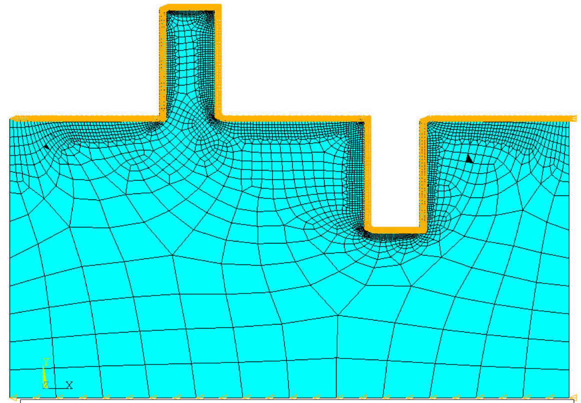
- The main purpose of the analysis was to find stress concentration close to notches during the heating-cooling process.
- Temperature applied at the upper surface, T(t), had been taken from experiments. Analysis were performed for three notch's radius R (R = 0.5, 1.0, 2.0 mm).

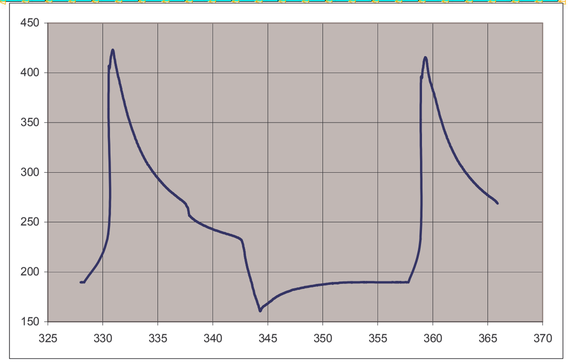
Initial and boundary conditions:

- Temperature of the bottom surface (line in 2D model) $T = T(t_0)$ is constant during the process.
- Reference temperature for structural analysis $T_{REF} = T(t_0)$.Material properties – functions of temperature. No initial stress. Plane strain, Transient analysis, Elasto-plastic material behaviour

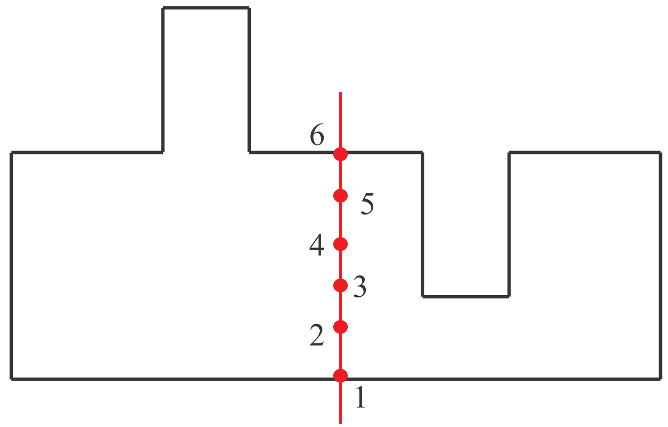


Dimensions of the analysed model

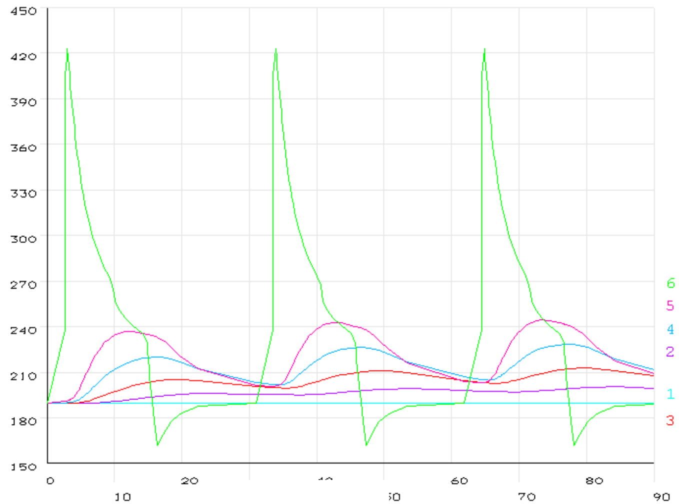


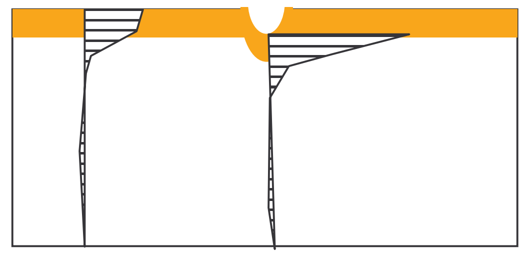


Temperature as the function of time at the surface of the die (from



Temperatures in points 1 to 6 during the process





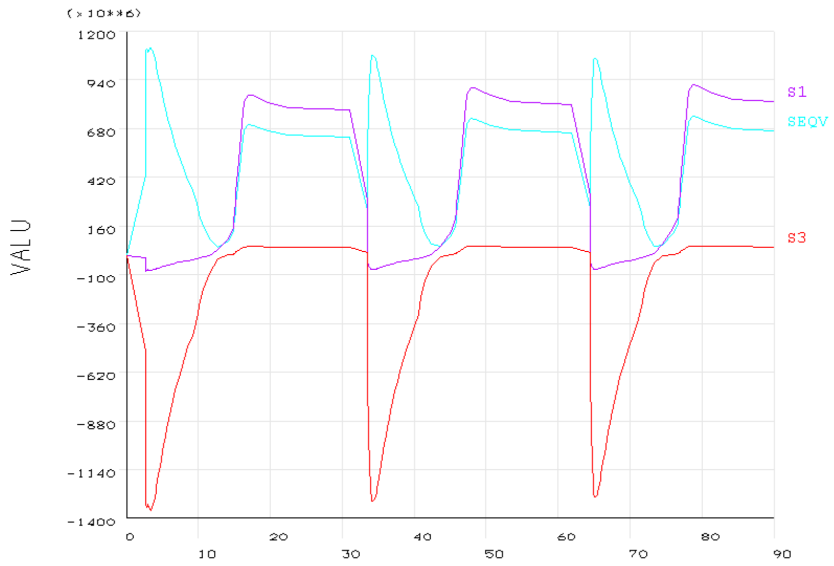
Model of the body with the thin layer subjected to heating- simple analytical considerations

Assuming h<< t we obtain within the heated layer

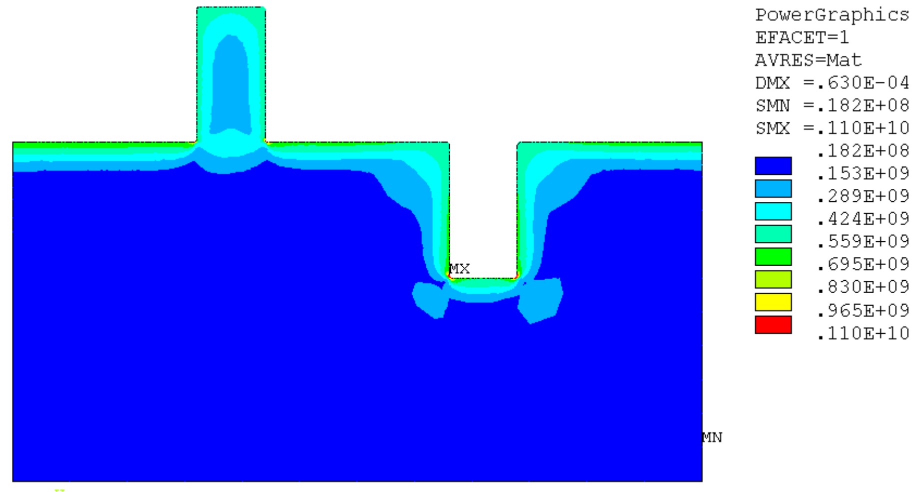
$$\varepsilon_x = 0,$$
$$\varepsilon_y = 0,$$
$$\sigma_x = \sigma_y = \frac{-1}{1-\nu} E\alpha\Delta T,$$
$$\sigma_z = 0.$$

For $\Delta T \cong 200C$ the result is $\sigma_x = \sigma_y = 685MPa$

$\overline{q} = -\lambda \,\mathrm{grad}(T) \qquad q = \alpha_k (T_b - T_s) \qquad Bi = \frac{\alpha_k \cdot l}{\lambda}$



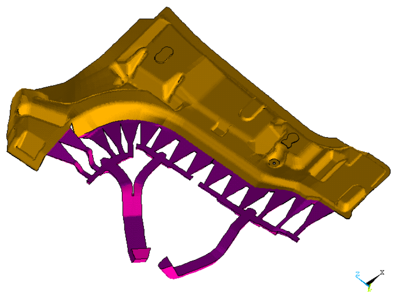
Von Mises (SEQV) and principal stresses at notch A. Notch radius R=0.5 mm



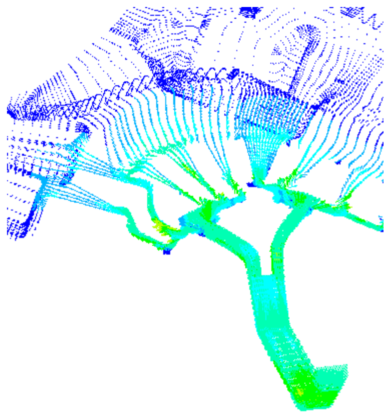
Von Mises stress distribution at time tx. Notch radius R=0.5 mm.

FE analysis of thin-walled elements' deformation during aluminium injection moulding

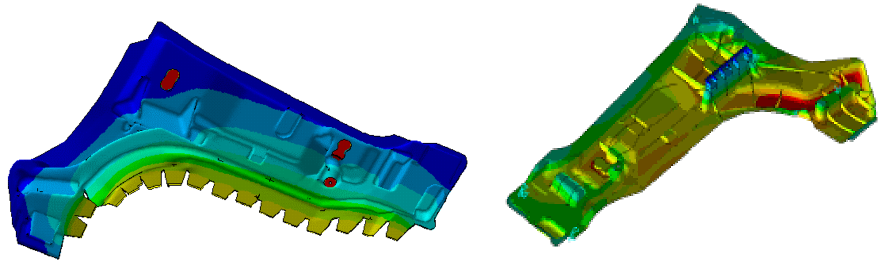
Numerical simulations have been performed to model the process of filling the mould by hot aluminium alloy. The analysis has enabled improvements of the element stiffness diminishing geometrical changes caused by the process. Fluid flow simulation with transient thermal analysis including phase change have been performed, followed by the structural elasto-plastic calculation of residual effects.



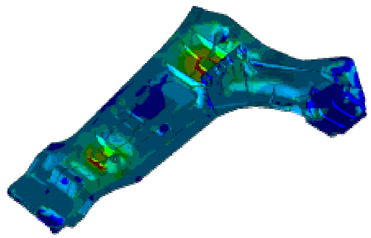
FE model



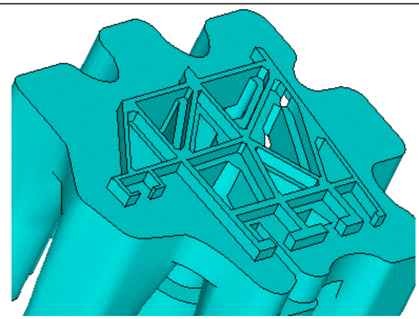
Velocity field during injection



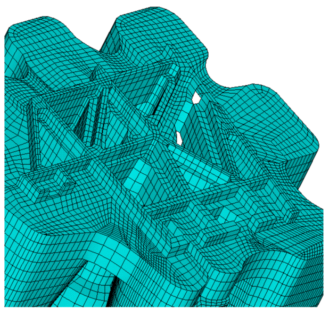
Temperature distribution (cooling effect) and displacements

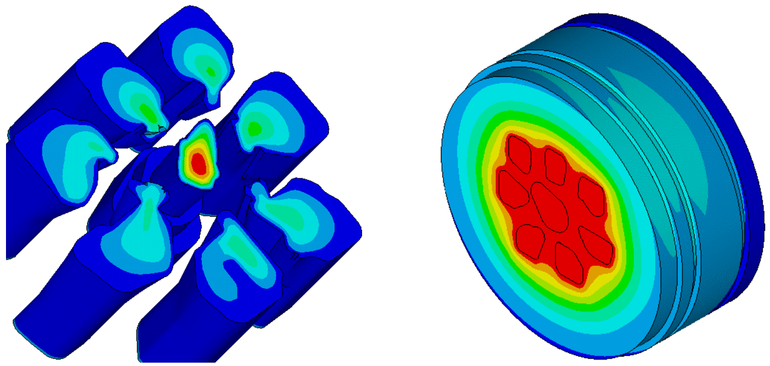


Residual stress distribution



FE model of the die





Velocity and temperature distribution

3. INTRODUCTION TO STRUCTURAL DYNAMICS

The transient dynamic equilibrium equation for a linear discrete structure (equations of motion):

$$\underset{n\times n}{\left[M\right]}\underset{n\times 1}{\{\overset{\circ\circ}{q}\}}+\underset{n\times n}{\left[C\right]}\underset{n\times 1}{\{\overset{\circ}{q}\}}+\underset{n\times n}{\left[K\right]}\underset{n\times 1}{\{q\}}=\underset{n\times 1}{\{F(t)\}},$$

$$\{\overset{\circ}{q}\}=\frac{d}{dt}\{q(t)\}=\begin{Bmatrix}\frac{dq_1(t)}{dt}\\ \frac{dq_2(t)}{dt}\\ \cdots\\ \frac{dq_n(t)}{dt}\end{Bmatrix},\qquad \{\overset{\circ\circ}{q}\}=\frac{d^2}{dt^2}\{q(t)\}=\begin{Bmatrix}\frac{d^2q_1(t)}{dt^2}\\ \frac{d^2q_2(t)}{dt^2}\\ \cdots\\ \frac{d^2q_n(t)}{dt^2}\end{Bmatrix}$$

where:

[M] = structural mass matrix [C] = structural damping matrix
[K] = structural stiffness matrix
{q} = nodal displacement vector , {q'} = nodal velocity vector
{q''} = nodal acceleration vector

{F(t)} = applied load vector

$$U=\frac{1}{2}\lfloor q\rfloor[K]\{q\}.\qquad T=\frac{1}{2}\lfloor\overset{\circ}{q}\rfloor[M]\{\overset{\circ}{q}\}$$

Rayleigh model of damping

$$\left[C\right]=\alpha_t\left[M\right]+\beta_t\left[K\right],$$

α_t β_t - external and internal damping coefficients

Special cases :

{F(t)} = {0} - free vibrations

{F(t)} = {0} [C]=[0] - free undamped vibrations (natural v.)

[M]=[0], {F(t)} = {0} [C]=[0] - linear static problem

FREE VIBRATIONS - modal analysis

$$[M]\{\ddot{q}\}+[K]\{q\}=\{0\}.$$

Second order set of differential equations

General solution $\{q(t)\}=\{q\}_A\cos\omega t+\{q\}_B\sin\omega t$,

$\{q\}_A$ i $\{q\}_B$ - vectors evaluated from the initial conditions.

ω - natural circular frequency

$$\{\ddot{q}\}=-\omega^2\{q\}_A\cos\omega t-\omega^2\{q\}_B\sin\omega t=-\omega^2\{q\}.$$

$$-\omega^2[M]\{q\}+[K]\{q\}=\{0\},$$

Finally : the **eigenvalue problem** : $\left([K]-\omega^2[M]\right)\{q\}=\{0\}$.

Trivial solution $\{q\}=\{0\}$

Nontrivial solutions $\det\left([K]-\omega^2[M]\right)=0$.

The determinant - polynomial of n-th degree in terms of ω^2. ω_i - natural frequencies (eigenvalues) $\{q\}_i$ -natural modes (eigenvectors)

normalisation $\lfloor q\rfloor_i[M]\{q\}_i=1$, $\lfloor q\rfloor_i[M]\{q\}_j=0,\quad i\neq j$. $\lfloor q\rfloor_i[M]\{q\}_j=\delta_{ij}$.

EIGENVALUES AND EIGENVECTORS IN ALGEBRA

Consider the special form of the linear system in which the right-hand side vector **y** is a multiple of the solution vector **x**:

$$\mathbf{A}\mathbf{x} = \lambda\, \mathbf{x}$$

or, written in full,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & \lambda x_1 \\ a_{21}x_2 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & \lambda x_2 \\ \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & \lambda x_n \end{array}$$

This is called the standard (or classical) ***algebraic eigenproblem***. The system can be rearranged into the homogeneous form

$$(\mathbf{A} - \lambda\mathbf{I})\, \mathbf{x} = \mathbf{0}.$$

A nontrivial solution of this equation is possible if and only if the coefficient matrix $\mathbf{A}-\lambda\mathbf{I}$ is singular.

Such a condition can be expressed as the vanishing of the determinant : $|\mathbf{A} - \lambda\mathbf{I}| = 0$

When this determinant is expanded, we obtain an algebraic polynomial equation in λ of degree n:

$$P(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \cdot \quad \cdot \quad \cdot + \alpha_n = 0$$

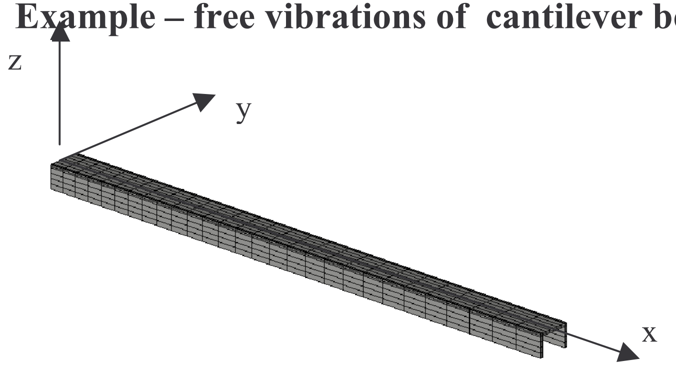
This is known as the ***characteristic equation*** of the matrix **A**. The left-hand side is called the ***characteristic polynomial***. We known that a polynomial of degree n has n (generally complex) roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. These n numbers are called the ***eigenvalues***, ***eigenroots*** or ***characteristic values*** of matrix **A**.

With each eigenvalue λi there is an associated vector $\mathbf{x}_i$ that satisfies

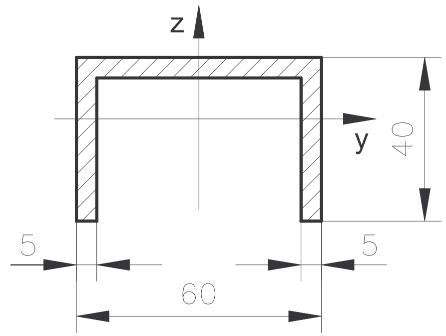
$$\mathbf{A}\mathbf{x}_i = \lambda_i\, \mathbf{x}_i.$$

This $\mathbf{x}i$ is called an ***eigenvector* or *characteristic vector***. An eigenvector is unique only up to a scale factor: if $\mathbf{x}_i$ is an eigenvector, so is $\beta\mathbf{x}_i$ where β is an arbitrary nonzero number. Eigenvectors are often *normalized* so that e.g. their Euclidean length is 1.

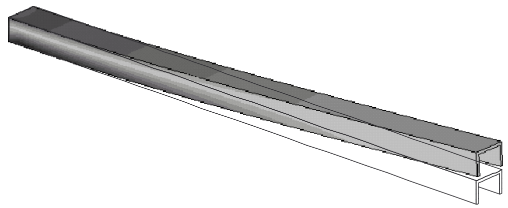
Example – free vibrations of cantilever beam, Mode shapes and natural frequencies



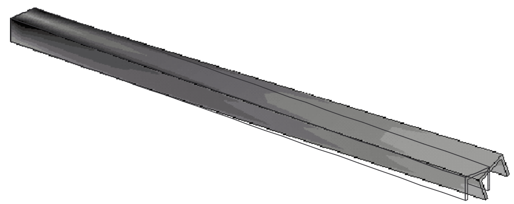
FE model



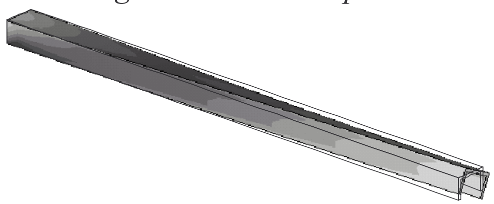
E=2*10^5 MPa
ν=0.3
$\rho=8*10^3 \frac{kg}{m^3}$



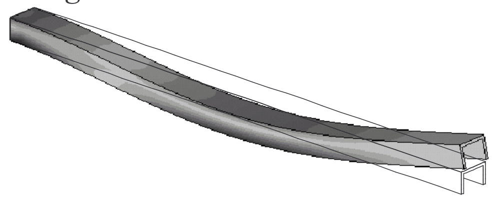
mode shape 1 , ω_1=219.1
endig vibrations in xz plane



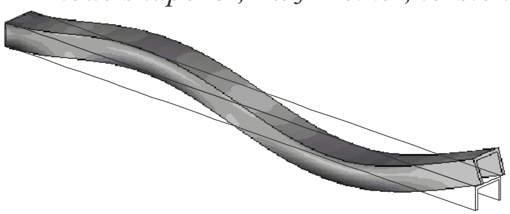
mode shape 2 , ω_2=333.3
Bending and torsional vibrations



mode shape 3 , ω_3=769.5 , torsional vibr



mode shape 4 ω_4=1353.73



mode shape 7 , ω_1=3704.3 , bending

Mass matrix of a finite element

$$T_e = \frac{1}{2}\lfloor \dot{q} \rfloor_e [m]_e \{\dot{q}\}_e .$$

Displacement vector within the element

$$\{u\} = [N]\{q\}_e ,$$

Velocity vector

$$\frac{d}{dt}\{u\} = [N]\{\dot{q}\}_e .$$

kinetic energy of the part $d\Omega_e$ of the finite element Ω_e equals

$$dT_e = \frac{1}{2}\lfloor \dot{u} \rfloor dm \{\dot{u}\} = \frac{1}{2}\lfloor \dot{u} \rfloor \rho \{\dot{u}\} d\Omega_e , \qquad \rho \text{ - density}$$

$$dT_e = \frac{1}{2}\lfloor \dot{q} \rfloor_e [N]^T \cdot \rho [N]\{\dot{q}\}_e d\Omega_e .$$

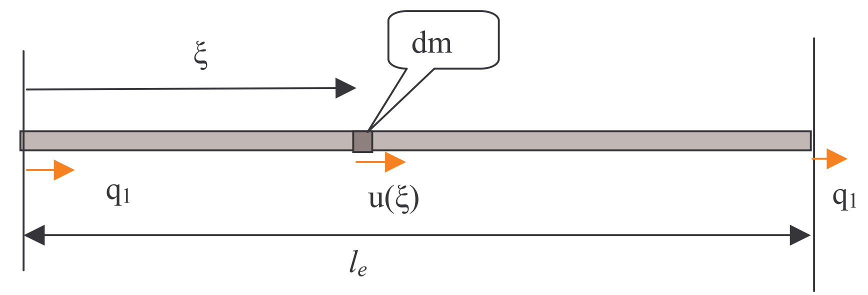
$$T_e = \frac{1}{2}\lfloor \dot{q} \rfloor_e \int_{\Omega_e} [N]^T \rho [N] d\Omega_e \{\dot{q}\}_e .$$

General formula for the consistent mass matrix of a finite element

$$[m]_e = \int_{\Omega_e} [N]^T \rho [N] d\Omega_e ,$$

$$T_e = \frac{1}{2}\lfloor \dot{q} \rfloor_e [m]_e \{\dot{q}\}_e .$$

Finite element formulation for rod element



The mass matrix for an axial member

$$T_e = \int_0^{l_e} \frac{dm(\dot{u})^2}{2} = \frac{1}{2}\int_0^{l_e} (\dot{u})^2 \rho A d\xi .$$

Velocity of the particles along the element

$$\dot{u}(\xi) = \lfloor N_1(\xi), N_2(\xi) \rfloor \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix},$$

$$N_1 = 1 - \frac{\xi}{l_e}, \qquad N_2 = \frac{\xi}{l_e},$$

ξ- local coordinate

Evaluating the integrals we get

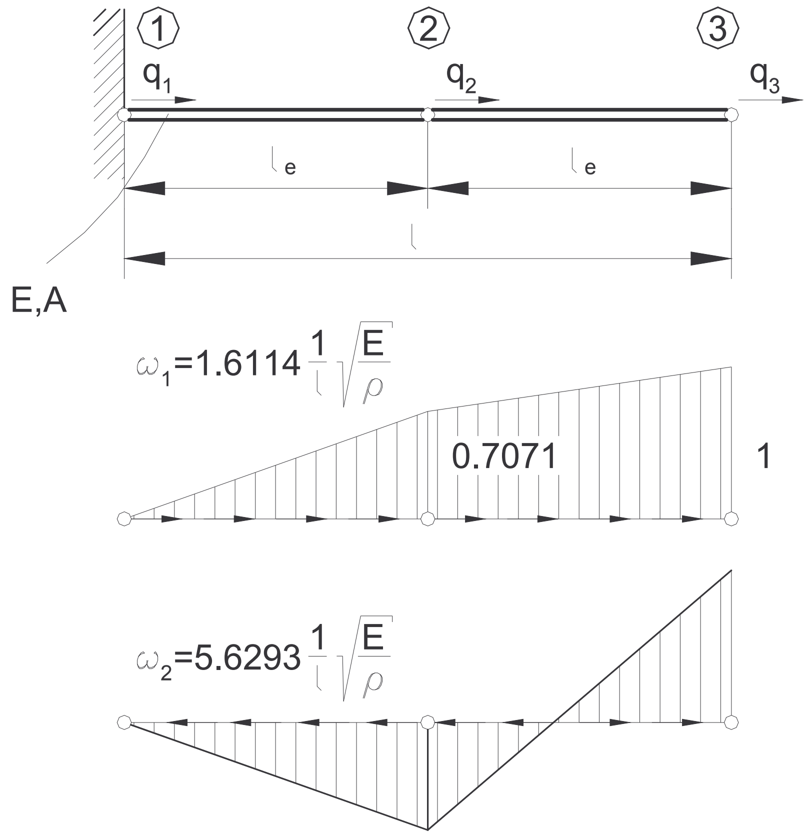
$$T_e = \frac{1}{2} \lfloor \dot{q}_1, \dot{q}_2 \rfloor [m]_e \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

$$[m_e] = \frac{\rho A l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Using diagonal (lumped) form of the matrix

$$[m]_e = \begin{bmatrix} \frac{\rho A l_e}{2} & 0 \\ 0 & \frac{\rho A l_e}{2} \end{bmatrix}.$$

Example



Free vibrations of the rod fixed at one end – FE model with 2 elements

<u>**The analytical solution**</u>

:

$$\omega_i^s = \frac{2i-1}{2}\pi\frac{1}{l}\sqrt{\frac{E}{\rho}}\,.$$

<u>FE solution using model with 2 finite elements</u>

$$[k]_e = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad [m]_e = \frac{\rho A l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad l_e = \frac{l}{2}.$$

Free vibrations equation

$$\left([K] - \omega^2 [M]\right)\{q\} = \{0\},$$

$$\left(\frac{EA}{l_e}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A l_e}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

$q_1 = 0$

$$\left(\frac{EA}{l_e}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A l_e}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Substituting $\lambda = \omega^2 \frac{\rho A l_e}{6} \Big/ \frac{EA}{l_e} = \frac{\rho l_e^2}{6E}\omega^2$ we have

$$\det\begin{bmatrix} 2-4\lambda & -(1+\lambda) \\ -(1+\lambda) & (1-2\lambda) \end{bmatrix} = 0.$$

Ihe roots of the characteristic equation of the matrix:
$\lambda_1 = 0.1082$, $\lambda_2 = 1.3204$ and consequently

$$\omega_1 = 0.8057\frac{1}{l_e}\sqrt{\frac{E}{\rho}} = 1.6114\frac{1}{l}\sqrt{\frac{E}{\rho}},$$

$$\omega_2 = 2.8148\frac{1}{l_e}\sqrt{\frac{E}{\rho}} = 5.6293\frac{1}{l}\sqrt{\frac{E}{\rho}}.$$

Comparing with the exact solution

$$\omega_1^s = 1.5708\frac{1}{l}\sqrt{\frac{E}{\rho}},$$

$$\omega_2^s = 4.7124\frac{1}{l}\sqrt{\frac{E}{\rho}},$$

we have the relative errors of the natural frequencies 2.6% and 19.5%.

<u>Mode shapes (eigenvectors</u>

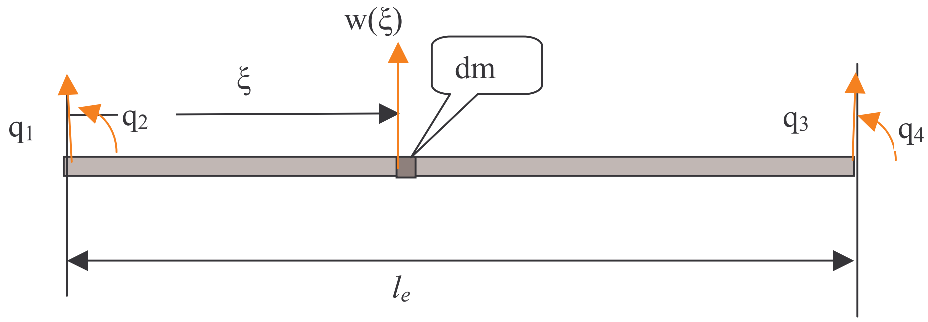
Assuming $q_3 = \Delta$ and $q_1 = 0$.

$$\lfloor q \rfloor_1 = \lfloor 0, \quad 0.707\Delta, \quad \Delta \rfloor,$$

$$\lfloor q \rfloor_2 = \lfloor 0, \ -0.707\Delta, \quad \Delta \rfloor.$$

One-dimensional beam element

Kinetic energy of the segment $d\xi$ of the beam



$$dT_e = dm \cdot (\dot{w})^2 / 2 \quad \text{(without the rotational movement)}$$

Velocity of the segment

$$\dot{w}(\xi) = \lfloor N_1(\xi), N_2(\xi), N_3(\xi), N_4(\xi) \rfloor \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{Bmatrix},$$

N_i - shape functions of the beam element

Kinetic energy of the element :

$$T_e = \int_0^{l_e} dT_e = \frac{1}{2}\int_0^{l_e} (\dot{w})^2 \rho A d\xi .$$

$$T_e = \frac{1}{2} \lfloor \dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4 \rfloor \cdot \frac{\rho A l_e}{420} \begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ & 4l_e^2 & 13l_e & -3l_e^2 \\ \text{sym.} & & 156 & -22l_e \\ & & & 4l_e^2 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{Bmatrix}$$

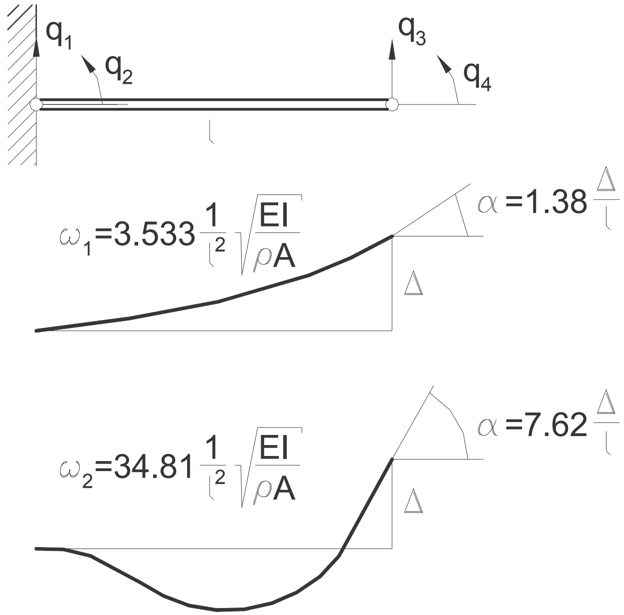
The mass matrix

$$[m]_e = \frac{\rho A l_e}{420} \begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ & 4l_e^2 & 13l_e & -3l_e^2 \\ & & 156 & -22l_e \\ & & & 4l_e^2 \end{bmatrix}$$

The same result may be derived using the general formula $[m]_e = \int_{\Omega_e} [N]^T \rho [N] d\Omega_e$,

Example - Natural frequencies of the cantilever beam .

One element FE model



The exact analytical solution

$$\omega_1^s = 3.5156 \cdot \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}},$$

$$\omega_2^s = 22.0346 \cdot \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}},$$

$$\omega_i^s = \left[\frac{(2i-1)\pi}{2}\right]^2 \cdot \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}, \qquad i = 3, 4...,$$

Matrix eigenvalue problem

$$\left(\frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ & 2l^2 & -3l & l^2 \\ & & 6 & -3l \\ & & & 2l^2 \end{bmatrix} - \omega^2\frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ & & & 4l^2 \end{bmatrix}\right)\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}.$$

$q_1 = 0$, $q_2 = 0$

$$\left(\frac{2EI}{l^3}\begin{bmatrix} 6 & -3l \\ -3l & 2l^2 \end{bmatrix} - \frac{\omega^2 \rho A l}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\right)\begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det\left(\begin{bmatrix} 6 & -3l \\ -3l & 2l^2 \end{bmatrix} - \lambda\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\right) = 0,$$

Introducing the new parameter

$$\lambda = \frac{\rho A l^4}{840EI}\cdot\omega^2.$$

we obtain the characteristic equation

$$140\lambda^2 - 204\lambda + 3 = 0,$$

and the roots

$$\lambda_1 = 1.4857\cdot 10^{-2},$$
$$\lambda_2 = 1.4423.$$

Consequently

$$\omega_1 = 3,533\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}},$$
$$\omega_2 = 34,81\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}.$$

Eigenvectors

$$\begin{bmatrix} 6-156\lambda & -3l+22l\lambda \\ -3l+22l\lambda & 2l^2-4l^2\lambda \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

The vectors $\lfloor q \rfloor_1 = \lfloor q_3, q_4 \rfloor_1$ and $\lfloor q \rfloor_2 = \lfloor q_3, q_4 \rfloor_2$ corresponding to ω_1 i ω_2 (λ_1 i λ_2).

$$q_4 = \frac{-(6-156\lambda)}{(-3+22\lambda)} \cdot \frac{q_3}{l} \quad \textbf{\textit{or}} \quad q_4 = \frac{-3+22\lambda}{-2+4\lambda} \cdot \frac{q_3}{l}.$$

Assuming $q_3 = \Delta$ we obtain for the first mode $q_4 = 1.38\frac{\Delta}{l}$, and for the second mode $q_4 = 7,62 \cdot \frac{\Delta}{l}$.

$$\lfloor q \rfloor_1 = \left\lfloor 0, 0, \Delta, 1.37\frac{\Delta}{l} \right\rfloor,$$

$$\lfloor q \rfloor_2 = \left\lfloor 0, 0, \Delta, 7.62\frac{\Delta}{l} \right\rfloor.$$

MODAL ANALYSIS by FEM : Good accuracy even for a poor discretization.

The best accuracy for the lowest frequencies

Summary- types of dynamic analyses in FEM:

Transient Dynamic

All dynamic analysis types in the ANSYS program are based on the following general equation of motion for a finite element system:

$$[M]\{\ddot{u}\} + [C]\{\mathring{u}\} + [K]\{u\} = \{F(t)\}$$

where:

[M] mass matrix , [C] damping matrix , [K] stiffness matrix

$\{\ddot{u}\}$ nodal acceleration vector, $\{\mathring{u}\}$ nodal velocity vector, $\{u\}$ nodal displacement vector

$\{F\}$ load vector , (t) time

Transient dynamic analysis (also known as time-history analysis) is used to determine the dynamic response of a structure subjected to time-dependent loads. There are three basic methods of a transient dynamic solution: full transient dynamic method, reduced method, and mode superposition..

The full transient dynamic is the most general. It has full nonlinear capability and may include plasticity, creep, large deflection, large strain, stress stiffening, and nonlinear elements.

Modal

Modal analysis is useful for any application in which the natural frequencies of a structure are of interest

For example, a machine component should be designed to produce natural frequencies that will prevent the component from vibrating at one of its fundamental modes under operating conditions.

Modal analysis is used to extract the natural frequencies and mode shapes of a structure. It is important as a first step to any dynamic analysis because knowledge of the structure's fundamental mode shapes and frequencies can help characterize its dynamic response. Some transient and harmonic solution procedures require the results of a modal analysis.

For undamped cases (which are most common for modal analysis) the damping term, $[C]\{\mathring{u}\}$, is ignored and the equation reduces to:

$$([K] - \omega^2[M])\{u\} = 0$$

where ω^2 (the square of natural frequencies) represents the eigenvalues, and $\{u\}$ represents the eigenvectors (the mode shapes, which do not change with time).

Harmonic Response

Harmonic response analysis is used to determine the steady-state response of a linear structure to a sinusoidally varying forcing function. This analysis type is useful for studying the effects of load conditions that vary harmonically with time, such as those experienced by the housings, mountings, and foundations of rotating machinery.

Response Spectrum

A response spectrum analysis can be used to determine the response of a structure to shock loading conditions.

This analysis type uses the results of a modal analysis along with a known spectrum to calculate maximum displacements and stresses that occur in the structure at each of its natural frequencies. A typical response spectrum application is seismic analysis, which is used to study the effects of earthquakes on structures such as piping systems, towers, and bridges.

Random Vibration

Random vibration analysis is a type of spectrum analysis used to study the response of a structure to random excitations, such as those generated by jet or rocket engines.

4. NONLINEAR STRUCTURAL ANALYSIS

LINEAR FEM MODEL OF DEFORMABLE STRUCTURE BEHAVIOUR:

$$[K]\{q\} = \{F\},$$

$$\{q\} = [K]^{-1}\{F\}.$$

Superposition method:

$$\{F_*\} = \alpha\{F_a\} + \beta\{F_b\}$$

$$\{q_*\} = [K]^{-1}\{F_*\},$$

$$\{q_*\} = [K]^{-1}(\alpha\{F_a\} + \beta\{F_b\}) = \alpha[K]^{-1}\{F_a\} + \beta[K]^{-1}\{F_b\}$$

$$\{q_*\} = \alpha\{q_a\} + \beta\{q_b\}$$

Structural nonlinearities cause the response of a structure to vary disproportionately with the applied forces. Realistically, almost all structures are nonlinear in nature but not always to a degree that the nonlinearities have a significant effect on an analysis.

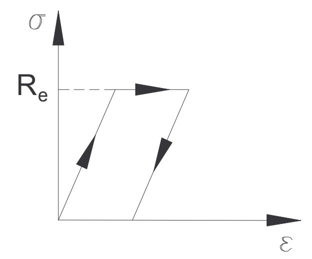
NONLINEAR FEM MODEL

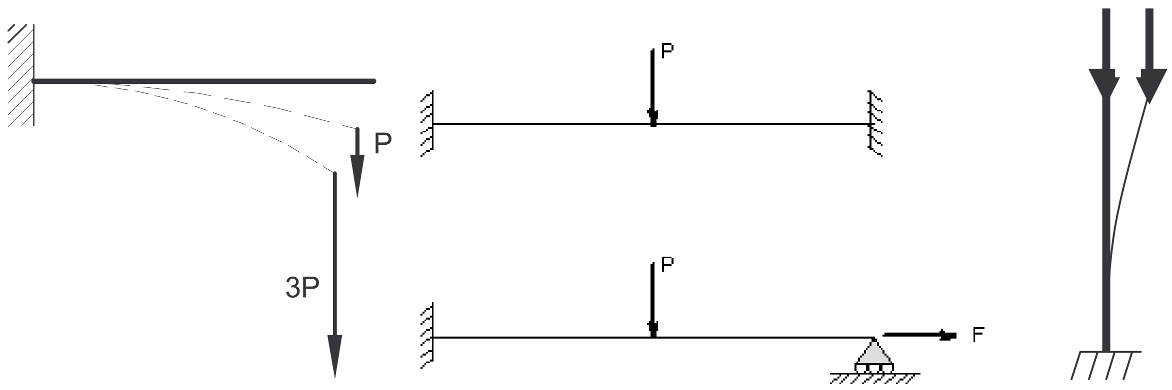
In a nonlinear analysis, the structure's stiffness matrix and load vector may depend on the solution and therefore are unknown. To solve the problem, the program uses an iterative procedure in which a series of linear approximations converges to the actual nonlinear solution.

$$\left[K(\{q\})\right]\{q\} = \{F(q)\}.$$

- If the solution exists? How many solutions exist?
- Time consuming solution
- Iterative process of solution – problem of convergence
- Results of a load depend on loading history

Causes of Nonlinear Behaviour





simple elasto-plastic model of material behaviour

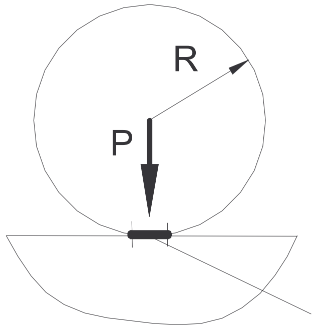
large deformations of structures

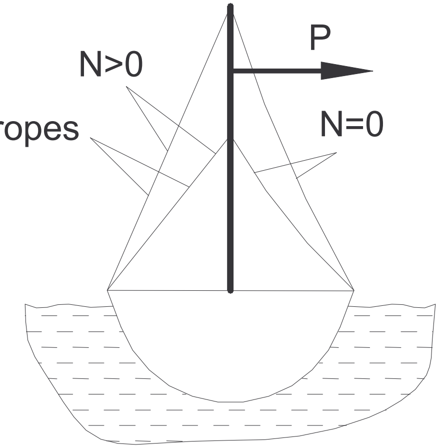
Material Nonlinearities *- nonlinear stress – strain relationships. Many factors can influence a material's stress-strain properties, including load history (as in elasto-plastic response) environmental conditions (such as temperature), and the amount of the time that a load is applied (as in creep response)*

Geometric Nonlinearities. If a structure experiences large deformations, its changing geometric configuration can cause the structure to respond nonlineary.

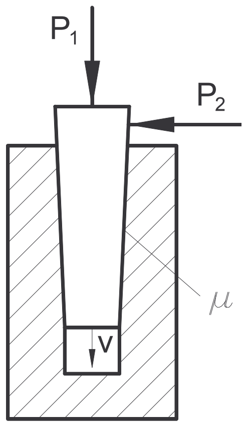
Under lateral loads, the beam is very flexible. As the force P increases, the rod deflects so much that the moment arm decreases appreciably, causing the increasing stiffness at higher loads. Axial forces can increase or decrease the stiffness of the beam depending oh the direction of the forces.

Friction, contact interaction, gaps, ropes

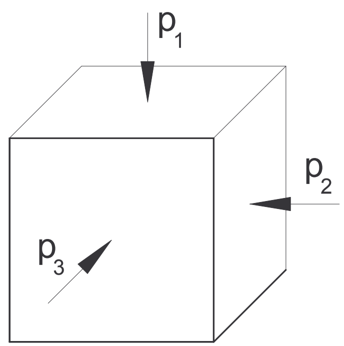




Importance of the history of loading on the result of the final load



friction



elasto-plastic material

Iterative solution of a set of nonlinear simultaneous equations

The series of approximate solutions (iterations) $\{q\}_0$, $\{q\}_1$, $\{q\}_2$, ..., $\{q\}_n$

converging to the exact solution

.The vector $\{q\}_i$ is calculated on the base of the previous solution $\{q\}_{i-1}$:

$$\left[K\left(\{q\}_{i-1}\right)\right]\{q\}_i = \{F\}.$$

$\{q\}_0$ – arbitrary initial solution (=0),

$$[K]_i = \left[K\left(\{q\}_i\right)\right].$$

Convergence criteria

DOF increment convergence $\{\Delta q\}_i = \{q\}_i - \{q\}_{i-1}$,

Out of balance convergence (residual vector)

$$\{R\}_{i+1} = \{F\} - [K]_i \{q\}_i ,$$

$$\left\|\{\Delta q\}_i\right\| \le \delta ,$$

$$\left\|\{R\}_i\right\| \le \varepsilon ,$$

δ i ε - the reference values

Norms:

$$\left\|\{x\}\right\| = \left(\lfloor x \rfloor \{x\}\right)^{\frac{1}{2}} ,$$

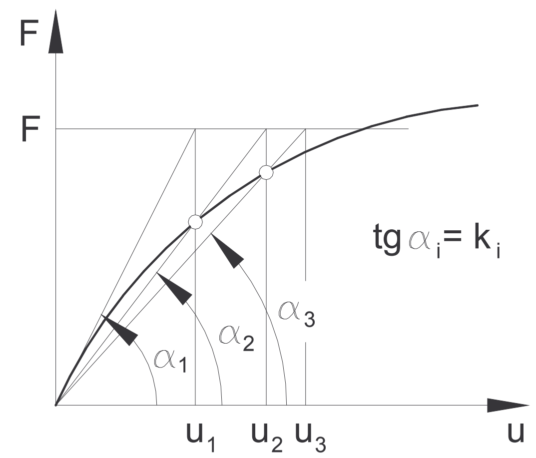
$$\left\|\{x\}\right\| = \max x_i ,$$

Criteria concerning the relative errors

$$\frac{\left\|\{\Delta q\}_i\right\|}{\left\|\{q\}_i\right\|} \le \varepsilon ,$$

$$\frac{\left\|\{R\}_i\right\|}{\left\|\{F\}\right\|} \le \delta .$$

Direct approach

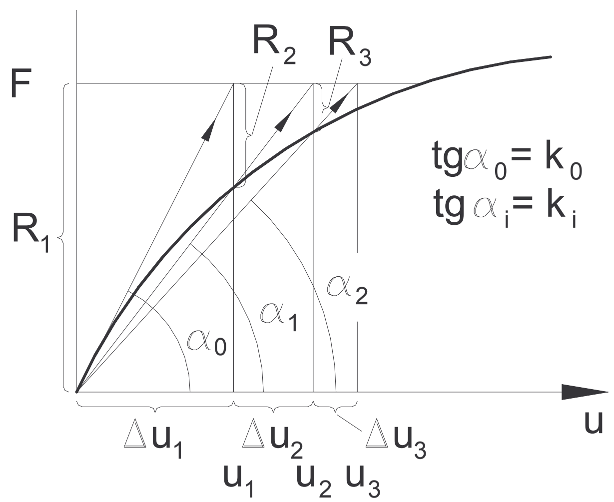


$$\{q\}_i = [K]_{i-1}^{-1}\{F\}$$

Incremental approach

The calculations concern increments of the unknowns vector $\{q\}_i$

Steps:



$$\{R\}_i = \{F\} - [K]_{i-1}\{q\}_{i-1},$$

Increment

$$\{\Delta q\}_i = [K]_{i-1}^{-1}\{R\}_i .$$

Tne new approximate solution and the matrix

$$\{q\}_i = \{q\}_{i-1} + \{\Delta q\}_i ,$$

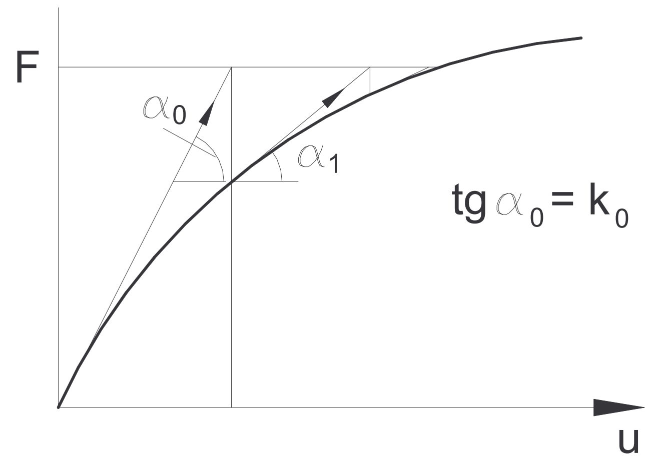
$$[K]_i = \left[K\left(\{q\}_i\right)\right],$$

Newton-Raphson method

In each itaration in the calculations of linear set of the equation uses the <u>tangent matrix</u>:

$$[K]_T = \frac{d\{F\}}{d\{q\}} = [K] + \frac{d[K]}{d\{q\}}\{q\}$$

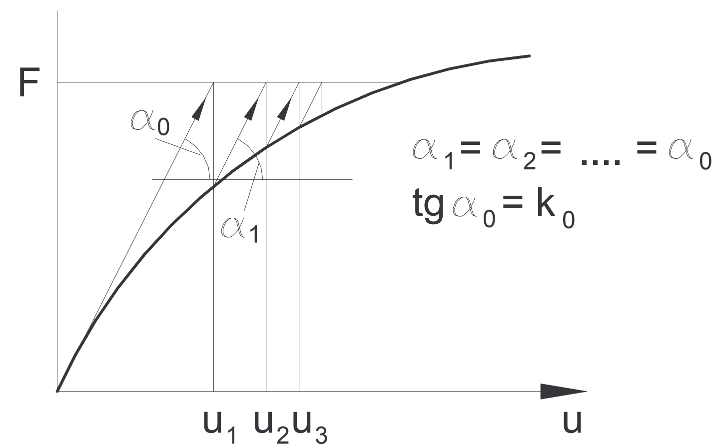
instead of coefficient matr



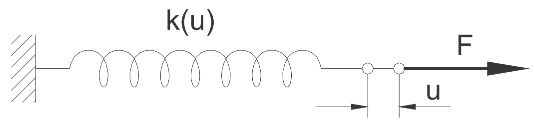
Modified Newton–Raphson procedure

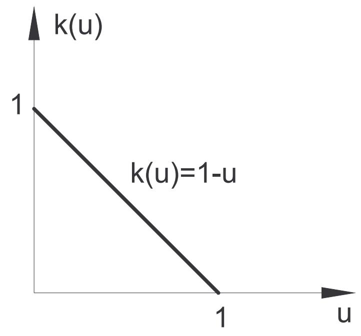
In each iteration the same set of the equations (the same <u>initial matrix</u>) are used

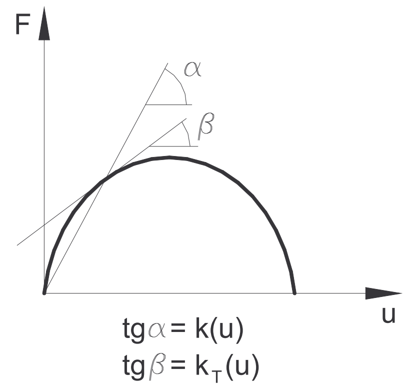
$$[K_0]^{-1} \text{ instead of } [K]_{i-1}^{-1}$$



Example:







Find the displacement u for the nonlinear spring:

$$k(u) = 1 - u$$
$$F = 0.2$$

Analytical solution

$$k(u)u = F\,, \quad u^2 - u + F = 0$$
$$u_1 = \frac{1-\sqrt{1-4F}}{2} = 0.2734,$$
$$u_2 = \frac{1+\sqrt{1-4F}}{2} = 0.7236.$$

tangent stiffness

$$k_T = \frac{dF}{du} = \frac{d}{du}\big(k(u)u\big) = \frac{dk}{du}u + k = 1 - 2u$$

Direct iteration procedure (the incremental approach

i	u_{i-1}	$k_{i-1} = 1 - u_{i-1}$	$R_i = F - k_{i-1}u_{i-1}$	$\Delta u_i = R_i / k_{i-1}$	$u_i = u_{i-1} + \Delta u_i$	$\frac{\Delta u_i}{u_i}$	$\frac{R_i}{F}$
1	0	1	0.2	0.2	0.2	1	1
2	0.2	0.8	0.04	0.05	0.25	0.2	0.2
3	0.25	0.75	0.0125	0.0167	0.2667	0.063	0.063
4	0.2667	0.733	0.0044	0.006	0.2727	0.022	0.022
5	0.2727	0.7273	0.0017	0.0023	0.2750	0.008	0.0085

Modified Newton-Raphson procedure

i	u_{i-1}	$k_{i-1} = 1 - u_{i-1}$	$R_i = F - k_{i-1}u_{i-1}$	$\Delta u_i = R_i / k_0$	$u_i = u_{i-1} + \Delta u_i$	$\frac{\Delta u_i}{u_i}$	$\frac{R_i}{F}$
1	0	1	0.2	0.2	0.2	1	1
2	0.2	0.8	0.04	0.04	0.24	0.167	0.2
3	0.24	0.76	0.0176	0.0176	0.2576	0.068	0.088
4	0.2576	0.7424	0.0087	0.00876	0.2664	0.033	0.044
5	0.2664	0.7336	0.0046	0.0046	0.2710	0.017	0.023
6	0.2710	0.729	0.0024	0.0024	0.2734	0.009	0.012

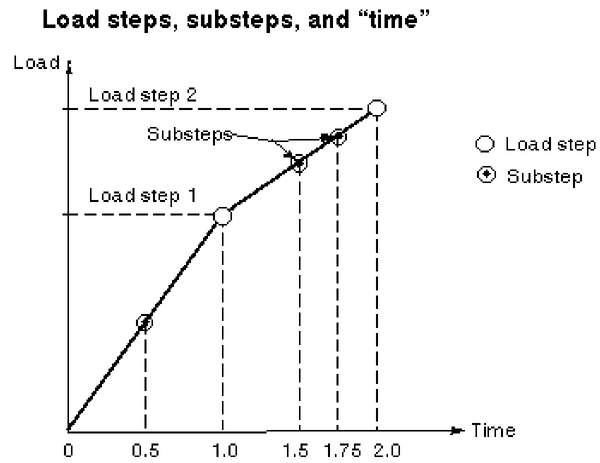
Newton-Raphson procedure

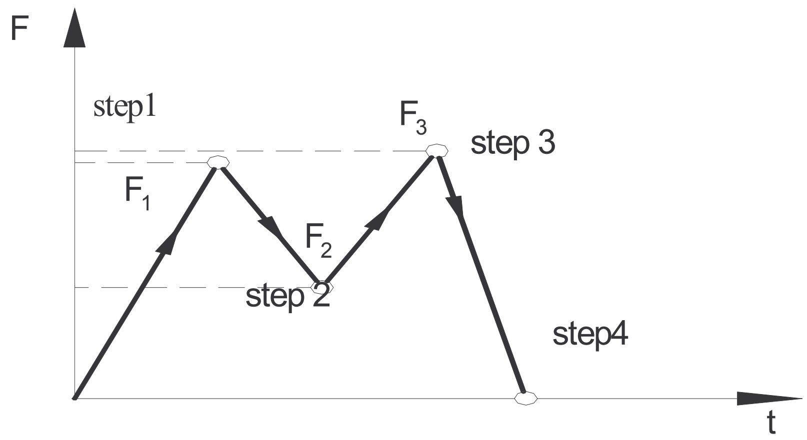
i	u_{i-1}	$k_{i-1} = 1 - u_{i-1}$	$R_i = F - k_{i-1}u_{i-1}$	$k_{Ti} = 1 - 2u_{i-1}$	$\Delta u_i = R_i / k_{Ti}$	$u_i = u_{i-1} + \Delta u_i$	$\frac{\Delta u_i}{u_i}$	$\frac{R_i}{F}$
1	0	1	0.2	1	0.2	0.2	1	1
2	0.2	0.8	0.04	0.6	0.0667	0.2667	0.250	0.2
3	0.2667	0.7333	0.0044	0.466	0.0095	0.2762	0.048	0.034
4	0.2762	0.7238	0.0001	0.448	0.0002	0.2764	0.001	0.0005

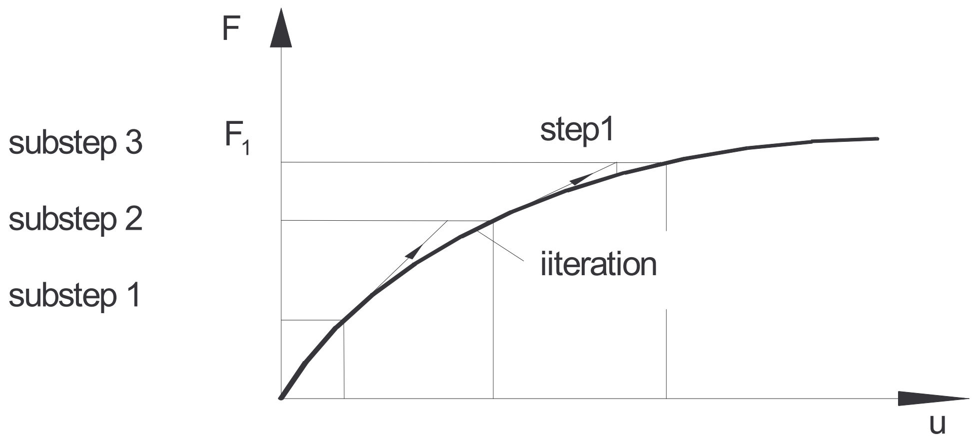
Iterative nonlinear calculations in practice

The user executes a nonlinear static analysis by subdividing the load into a series of incremental load steps and, at each step, performing a successive of linear approximations to obtain equilibrium.

Each linear approximation requires one pass through the equation solver (known as an equilibrium iteration).







5. PARAMETRIC FE MODELS AND DESIGN OPTIMISATION

The construction of a complex 3D FE model is not a straightforward task even when using the high-level and powerful programs.
The development of a large model may take days, and each modification represents a high investment.
In such cases parametric modeling seems to be natural..

Parametric approach permits a finite element model to be defined as a function of variables (parameters) instead of by the more conventional numerical data.
Dimensions can be expressed as named variables or expressions involving other dimensions. If we change a dimension later, the change will be automatically reflected in the entire model. Once the geometric model is ready, finite element meshing is automatic.

With parametric modeling, complete products lines can be analyzed using ONE MODEL.

Now, design studies or routine final product analyses can be completed with significant COST SAVINGS.

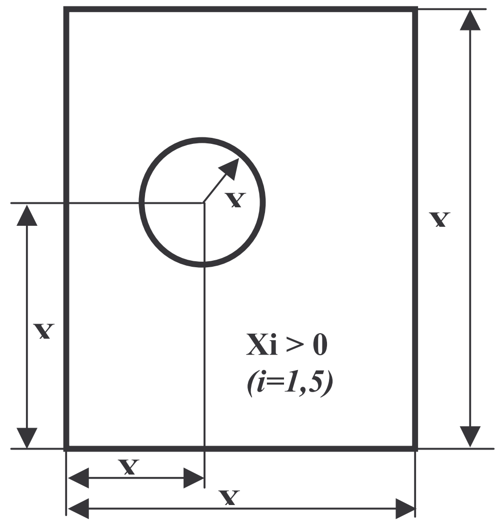
The example:
Rectangular plate with the circular hole

MODEL CREATION PROCESS

1. First the nominal topology of the object is created by means of ordinary geometric modeling or solid modeling operations. The result is a standard model exhibiting the desired geometric elements and connectivity between elements.

2. Next , some of the numbers used when building the model are replaced by the variables (parameters). T relationship between some of the parameters are described in terms of constraints.

3. New models may be created as variants of the basic model by changing the values of the constrained variables. After each change, a new instance of the model is created by re-executing the list of commands being the history of model creation or even the full analysis procedure.



Design optimization is a technique that seeks to determine an optimum design. By 'optimum desi fied requirements but with a minimum expense of certain factors such as weight, surface area, volume, stress, cost, etc.

Virtually any aspect of your design can be optimized: dimensions (such as thickness), shape (such as fillet radii), placement of supports, cost of fabrication, natural frequency, material property, and so on. Actually, any ANSYS item that can be expressed in terms of parameters can be subjected to design optimization.

The ANSYS program offers two optimization methods to accommodate a wide range of optimization problems. The *subproblem approximation* method is an advanced zero-order method that can be efficiently applied to most engineering problems. The *first order* method is based on design sensitivities and is more suitable for problems that require high accuracy.

For both the subproblem approximation and first order methods, the program performs a series of analysis-evaluation-modification cycles. That is, an analysis of the initial design is performed, the results are evaluated against specified design criteria, and the design is modified as necessary. This process is repeated until all specified criteria are met.

The standard optimum design problem formulation

Minimize the objective function $f = f(x)$

(eg. max. strain energy density, equivalent stress, max. principal strain)

where

$$x = (x_1, x_2, x_3 \ldots\ldots\ldots x_n)$$

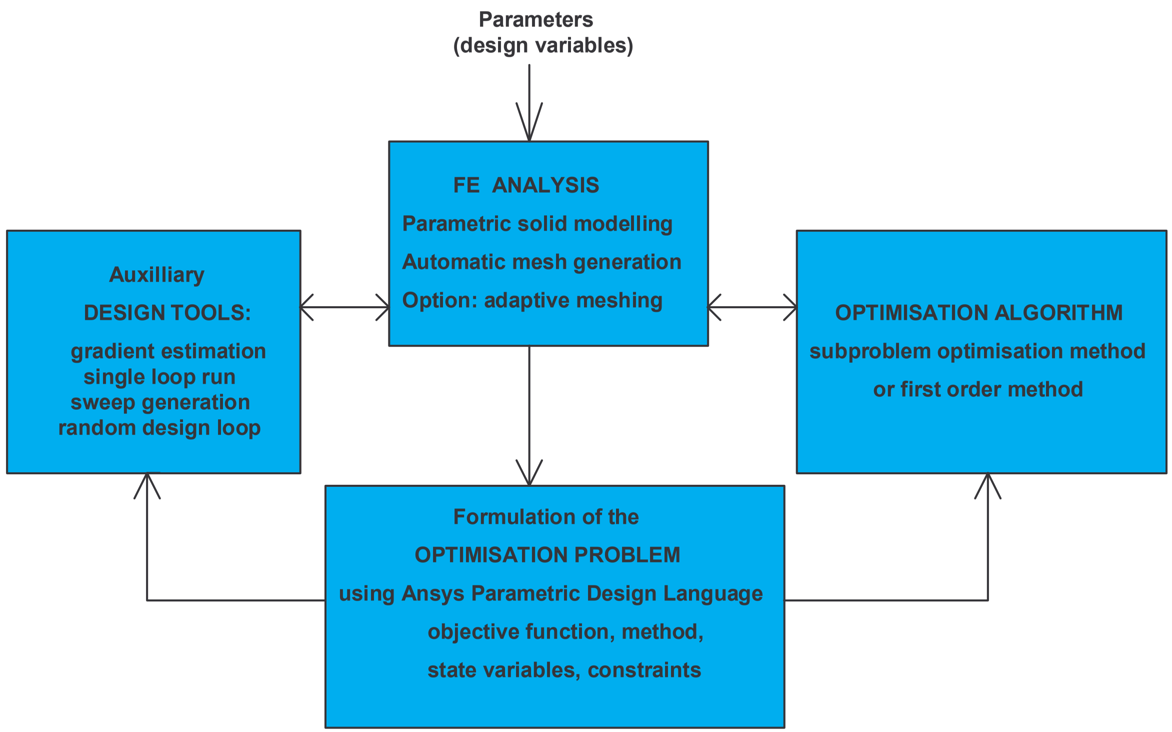
is the vector of design variables *(eg. material data, shape parameters)*

Constraints:

$$x_i^{*} \le x_i \le x_i^{**} \qquad (i=1,2....n)$$

$$w_i^{*} \le w_i(x) \le w_i^{**} \qquad (i=1,2....m)$$

Finite element modelling and optimization techniques



Developing of parametric FE models in mechanical engineering proved to be very useful for comparative analyses and design procedures.

The basic problem in parametric shape modeling is to avoid degeneracies of the solid model and ‘badly shaped’ elements , while performing automatic mesh generation. The important point is also the set of parameters chosen for shape representation.
The basic advantages in parametric approach is significant cost saving when building new variants of the model and easy comparative analyses. The parametric models can be easily used for design optimisation.

Problems in optimization : - convergence
- uniqueness
- constraints
- local or global minimum